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Fusion rules of the lowest weight representations of $\mathrm{osp}_q(1|2)$ at roots of unity: polynomial realization

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Abstract

The degeneracy of the lowest weight representations of the quantum superalgebra $\mathrm{osp}_q(1|2)$ and their tensor products at exceptional values of q is studied. The main features of the structures of the finite-dimensional lowest weight representations and their fusion rules are illustrated using realization of group generators as finite-difference operators acting in the space of the polynomials. The complete fusion rules for the decompositions of the tensor products at roots of unity are presented. The appearance of indecomposable representations in the fusions is described using Clebsh–Gordan coefficients derived for general values of q and at roots of unity.

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1. Introduction

Quantum algebras have been studied intensively from the very moment they were invented by Faddeev *et al* in 1981 [1]. Since then, they have found numerous applications in different fields of physics and mathematics and are related by thousands of links with other branches of science.

Being special, quantum algebras (or superalgebras) have in many cases the same representations as corresponding classical (non-deformed) Lie algebras, but in addition new, quite different representations appear in the quantum case [2–23]. In the standard deformation scheme (with dimensionless deformation parameter q) the center of the algebra is enlarged and new Casimir operators and correspondingly new types of representations appear when q is given by a root of unity [3–14, 16]. However, the structure of the representation space is not the only thing which is subjected to deformation: the decomposition of the tensor product of the representations is also deformed. The allowed values of 'spin' of the q-deformed finite-dimensional representations are restricted when the deformation parameter q is given by a root of unity. In this case, a proper subspace can appear inside the representation V, irreducible at general q, making the latter non-irreducible. As a consequence some items in the decomposition of the tensor products are unified into new, indecomposable representations $\mathcal{I}$ [3, 4, 7, 12]. Although there is a considerable amount of work (partly cited above) devoted to the representation theory at the roots of unity, it seems a thorough investigation of the fusion rules regarding all possible representations is needed (especially a detailed analysis of $V \otimes \mathcal{I}$ and $\mathcal{I} \otimes \mathcal{I}$).

In this paper, we clarify the aforementioned aspects in visual form for the lowest weight representations of the quantum superalgebra $\mathrm{osp}_q(1|2)$ [2, 4, 8–10, 15, 17, 18, 20]. The orthosymplectic superalgebras $\mathrm{osp}(n|2m)$ (and their quantum deformations) are actual in CFT, in the theory of integrable models, and in string theory (see the works [4, 21] and references therein). The change of the representation's spectrum at roots of unity brings to new peculiarities, for instance to new solutions of the Yang–Baxter equations in the theory of 2D integrable models [14, 16, 17].

There is a correspondence between the irreps of the graded algebra $\mathrm{osp}_q(1|2)$ and the irreps of the algebra $sl_{iq^{1/2}}(2)$ [4, 15, 17, 18] at general q, which allows us to use some techniques evolved from the study of the quantum deformation of $sl(2)$ [3, 4, 12, 22] for investigation of the superalgebra $\mathrm{osp}_q(1|2)$. Our approach consists of studying the finite-dimensional representations and their tensor products at general values of q, to find at the complex q-plane the 'singular points' (located on the unit circle) of these representations or their tensor products' decompositions.

The finite-difference realization of the group generators, acting on the space of the polynomials, provides a clear and compact description and provides the best solutions. The projection operator approach gives a simple understanding of why degeneracies appear in the decompositions of the representations' tensor products. The sufficiently comprehensible constructions presented here are adapted especially for physical applications.

The paper is organized as follows. In the second section, the algebra generators and the co-product in the aforementioned polynomial realization are presented and in the third section, some examples of the representations and their tensor products are considered in detail. The discussed patterns illustrate the principal cases, including multiple tensor products of the irreps and indecomposable representations. In the following section, an analysis of the fusion rules from the viewpoint of the projector operators is performed. The last sections are devoted to general analysis and conclusions based on the observations of the previous sections, accompanied by an explicit proof of the general formulae. In sections 5.1 and 5.2, we represent indecomposable representations emerging from the tensor product's decompositions of the odd-dimensional irreps, give a detailed computation of their dimensions, state the crucial principles on which they appear in the fusions and propose fusion rules for the decompositions. In section 5.3, Clebsh–Gordan coefficients in general form are derived, by means of which the fusion rules' degenerations can be presented by direct constructions (see the appendix). Section 6 is devoted to the even-dimensional representations and to the correspondence between the representations of the quantum deformations of the algebras $\mathrm{osp}(1|2)$ and $sl(2)$ at the exceptional values of the deformation parameter.

2. The algebra $\mathrm{osp}_q(1|2)$ and co-product: polynomial realization

The quantum algebra $\mathrm{osp}_q(1|2)$ [2, 4, 15] is a Hopf algebra. It is generated by two odd generators e, f and the even generators k, k^{-1}, which obey the following (anti-)commutation relations:

$$f k^{\pm 1} = q^{\pm 1} k^{\pm 1} f, \qquad e k^{\pm 1} = q^{\mp 1} k^{\pm 1} e, \qquad \{e, f\} = \frac{k - k^{-1}}{q - q^{-1}} = [H]_q, \tag{1}$$

where $q \in \mathbb{C}, q \neq 0, \pm 1, [a]_q = \frac{q^a - q^{-a}}{q - q^{-1}}$ and we use the notation $k^{\pm 1} = q^{\pm H}$ to maintain a connection with the non-deformed case. The bracket $\{,\}$ stands for the anti-commutator. When $q \to 1$ these relations reduce to the ordinary (anti-)commutation relations of the superalgebra osp(1|2) for the simple generators e, f, H.

The quadratic Casimir operator c, given by the formula

$$c = -(q + 2 + q^{-1})e^2 f^2 + (kq^{-1} + qk^{-1})ef + \left[H - \frac{1}{2}\right]_q^2$$
$$= \left((q^{\frac{1}{2}} + q^{-\frac{1}{2}})ef - \left[H - \frac{1}{2}\right]_q\right)^2, \tag{2}$$

is the square of a simpler operator, the so-called Scasimir operator [10, 11].

The algebra generators can be represented as finite-difference operators on the graded space of the polynomials:

$$k^{\pm 1} = q^{\pm(2x\partial + \theta\partial_\theta - 2j)} = q^{\pm(2x\partial - 2j)} + q^{\pm(2x\partial - 2j)}(q^{\pm 1} - 1)\theta\partial_\theta, \qquad f = \partial_\theta + \frac{\theta}{x}[x\partial]_q,$$
$$e = (x\partial_\theta + \theta[x\partial]_q)([x\partial + 1 - 2j]_q - [x\partial - 2j]_q) - [2j]_q\theta, \tag{3}$$

where θ is a Grassmann variable, while $x \in \mathbb{C}$. Note that $F = f^2 = D_q = \frac{1}{x}\frac{q^{x\partial} - q^{-x\partial}}{q - q^{-1}}$.

We fix the generator f to be a lowering operator and e to be a raising operator throughout the paper, as usual. In the polynomial representation there always exists a lowest weight vector, which is given by a constant function, i.e. the present method is especially convenient to study the lowest weight representations. For general values of q, the odd-dimensional lowest weight representations are in one-to-one correspondence with the representations of the non-deformed algebra osp(1|2) [2, 4, 8, 15] and can be classified in the same way. Spin-j $\left(j \in \frac{1}{2}\mathbb{Z}_+\right)$ representation with the eigenvalue $[2j + 1/2]_q^2$ of the Casimir operator c has dimension $4j + 1$. Degeneracy occurs and new features appear when q is given by a root of unity. The center of the algebra becomes larger. From the relations

$$[f, e^n\} = \begin{cases} e^{n-1}\left[H + \dfrac{n-1}{2}\right]_q \dfrac{[n]_q[1/2]_q}{[n/2]_q} & n \text{ is odd} \\ e^{n-1}\left(\left[H + \dfrac{n}{2}\right]_q - \left[H + \dfrac{n}{2} - 1\right]_q\right)\left[\dfrac{n}{2}\right]_q & n \text{ is even} \end{cases} \tag{4}$$

$$[f^n, e\} = \begin{cases} f^{n-1}\left[H + \dfrac{1-n}{2}\right]_q \dfrac{[n]_q[1/2]_q}{[n/2]_q} & n \text{ is odd} \\ f^{n-1}\left(\left[H - \dfrac{n}{2} + 1\right]_q - \left[H - \dfrac{n}{2}\right]_q\right)\left[\dfrac{n}{2}\right]_q & n \text{ is even} \end{cases} \tag{5}$$

it follows that if $q^N = 1$, operators $e^{\mathcal{N}}$, $f^{\mathcal{N}}$ and $k^{\mathcal{N}}$ commute with algebra generators, where $\mathcal{N} = 2N$ for odd N and $\mathcal{N} = N$ for even N. Here $[a, b\}$ is a graded commutator, which is an anti-commutator, when both a and b are odd operators, and is a commutator otherwise:

$$[a, b\} = ab - (-1)^{p(a)p(b)}ba.$$

$p(a)$ is the parity of the homogeneous element a of the graded algebra and equals to 0 for even (bosonic) elements and equals to 1 for odd (fermionic) elements.

The $\text{osp}(1|2)_q$ as a Hopf superalgebra possesses co-unit ϵ and antipode γ [15] defined as

$$\varepsilon(e) = 0, \qquad \varepsilon(f) = 0, \qquad \varepsilon(k) = 1, \qquad \varepsilon(k^{-1}) = 1,$$
$$\gamma(e) = -ek, \qquad \gamma(f) = -k^{-1}f, \qquad \gamma(k) = k^{-1}, \qquad \gamma(k^{-1}) = k. \tag{6}$$

The co-associative co-product compatible with (6) is given as follows:

$$\begin{aligned} \Delta(k) &= k \bar{\otimes} k, & \Delta(k^{-1}) &= k^{-1} \bar{\otimes} k^{-1} \\ \Delta(f) &= f \bar{\otimes} 1 + k \bar{\otimes} f, & \Delta(e) &= e \bar{\otimes} k^{-1} + 1 \bar{\otimes} e. \end{aligned} \tag{7}$$

Here $\bar{\otimes}$ denotes the supertensor product for the graded operators. On the product of two graded spaces of the polynomials V^i, $i = 1, 2$, generated by variables x_i, θ_i, the graded nature of the superalgebra is taken into account automatically via the inclusion of the Grassmann variables (see (3)):

$$k = k_1 k_2, \qquad k^{-1} = k_1^{-1} k_2^{-1} \qquad f = f_1 + k_1 f_2, \qquad e = e_1 k_2^{-1} + e_2, \tag{8}$$

where $\{k_i, k_i^{-1} e_i, f_i\}$ are the generators (3) acting on the spaces V_i correspondingly.

For the homogeneous elements a_i, c_i of the algebra the multiplication law for the graded tensor products [23] is

$$(a_1 \bar{\otimes} c_1)(a_2 \bar{\otimes} c_2) = (-1)^{p(c_1)p(a_2)}(a_1 a_2 \bar{\otimes} c_1 c_2). \tag{9}$$

From the above relation and the co-product (7) the following equations can be derived

$$\Delta(f^n) = \sum_{r=0}^{n} [{}^n_r]_{-q^{-1}} (f \bar{\otimes} 1)^{n-r} (k \bar{\otimes} f)^r, \qquad \Delta(e^n) = \sum_{r=0}^{n} [{}^n_r]_{-q^{-1}} (1 \bar{\otimes} e)^{n-r} (e \bar{\otimes} k^{-1})^r. \tag{10}$$

Here $[{}^n_r]_{-q^{-1}}$ are q-binomial coefficients [23]

$$[{}^n_r]_q = \frac{[n]_{q^{1/2}}! q^{(n-r)r/2}}{[r]_{q^{1/2}}! [n-r]_{q^{1/2}}!}, \tag{11}$$

with q-factorials $[p]_q! = [p]_q [p-1]_q \cdots [1]_q$.

If the superalgebra elements a, c have the matrix representations a_i^j, c_i^j in the representation spaces V and U respectively, which have basis states v_i and u_i, then the matrix representation of $(a \bar{\otimes} c)$ in the representation space $V \bar{\otimes} U$ with basis states $v_i \bar{\otimes} u_j$ is

$$(a \bar{\otimes} c)_{ij}^{kr} = a_i^k c_j^r (-1)^{p(k)(p(j)+p(r))}, \qquad p(i') = 0, 1; \quad i' = i, j, k, r. \tag{12}$$

Here $p(i')$ is the parity of the i'th basis element. In the later discussion for simplicity we shall use the usual notation $\otimes$ for the tensor product of the graded representations.

To establish the correspondence between the polynomial realization and the matrix formulation it is enough to assign the following columns to the vectors of the fundamental multiplet $\{1, \theta, x\}$: $1 = (0, 0, 1)^\tau$, $\theta = (0, 1, 0)^\tau$, $x = (1, 0, 0)^\tau$ (τ stands for transposition operation).

We are going to study the lowest weight representations, arising from the tensor products of the fundamental representations, and for them, in the case of q being a root of unity, e and f are $\mathcal{N}$-nilpotent: $e^{\mathcal{N}} = 0$, $f^{\mathcal{N}} = 0$ (it follows from (4), (7) and (10)) and from the existence of the lowest and highest weight vectors), and $k^{\mathcal{N}} = 1$. Note that in the case when $q^N = -1$ and N is an odd number, it is evident from (4) that the operators $e^N, f^N, k^{\pm N}$, similar to the Scasimir operator, anti-commute with the part of the algebra generators and commute with the other part. However, in the mentioned representation spaces, of interest to us, the operators e^N, f^N ($q^N = -1$, N is odd) also can be regarded as Casimir operators with 0 eigenvalues (here $k^N = \pm 1$).

The lowest weight representations at general q. One can see that the number of bosonic states in the $(2n+1)$-dimensional irrep $\{1, \theta, x, \ldots, x^{n-1}\theta, x^n\}, n \in \mathbb{N}$, exceeds the number of fermionic states by one (as the lowest weight vector 1 is bosonic). The quantum algebra $\mathrm{osp}_q(1|2)$ possesses the 'supersymmetric' even-dimensional representations, $\{1, \theta, x, \ldots, x^{n-1}\theta, x^n, x^n\theta\}$, with equal number of fermionic and bosonic basis states as well. Indeed, here it is necessary to have the fermionic vector $x^n\theta$ at some $n \in \mathbb{N}$ as a highest weight vector and one demands

$$e \cdot x^n\theta = 0 \qquad \text{or} \qquad [2j-n]_q - [2j-n-1]_q = 0. \tag{13}$$

Equation (13) has solutions, when $q^{4j} = -q^{2n+1}$. In other words, $r = (2n+2)$-dimensional irreducible representations form a sequence labelled by positive integer r or by j_r (see [15]):

$$2j_r = n + \frac{1}{2} + \lambda = \frac{r-1}{2} + \lambda, \qquad q^\lambda = \mathrm{i}, \qquad \lambda = \frac{\mathrm{i}\pi}{2\log q}. \tag{14}$$

So one sees that this series of representations has no classical counterpart.

The action of the algebra elements (3) on the states x^p, $x^p\theta$ of the spin-j irrep reads

$$\begin{aligned} &e \cdot x^p = ([p]_q([p+1-2j]_q - [p-2j]_q) - [2j]_q)x^p\theta, \\ &e \cdot x^p\theta = ([p+1-2j]_q - [p-2j]_q)x^{p+1}, \\ &f \cdot x^p = [p]_q x^{p-1}\theta, \qquad f \cdot x^p\theta = x^p \\ &k^{\pm 1} \cdot x^p = q^{\pm(2p-2j)}x^p, \qquad k^{\pm 1} \cdot x^p\theta = q^{\pm(2p-2j+1)}x^p\theta. \end{aligned} \tag{15}$$

The decomposition rule of the tensor product of two irreps with dimensions r_1 and r_2 is obtained in the same way as for the non-deformed algebra and can be proved by straight construction [15] (see section 4),

$$V_{r_1} \otimes V_{r_2} = \bigoplus_{r=|r_1-r_2|+1}^{r_1+r_2-1} V_r, \qquad \Delta r = 2. \tag{16}$$

The even-dimensional representations were described for the first time in [15], and as stated in this work, the representations can be defined up to the sign of the power index of the eigenvalues of the generator k, due to an automorphism of the algebra $(e \to -e,\ f \to f, k \to 1/k)$.

3. Fusion rules of the low-dimensional representations

In this section, we shall consider some simple examples to illustrate the main phenomena: the Clebsh–Gordan decomposition of tensor products depends on the deformation parameter and when it takes exceptional values the direct sum decomposition turns into the semi-direct one in certain cases. Then the block-diagonal action of the algebra generators on the tensor product becomes a block-triangular one. Let us start with the simplest case.

$\mathbf{V}_2 \otimes \mathbf{V}_2$. The tensor product of two such irreps, $\{1, \theta_1\} \otimes \{1, \theta_2\} = \{1, \theta_2, \theta_1, \theta_1\theta_2\}$, is decomposed into the direct sum of the states $\left\{1, \theta_1 - \mathrm{i}q^{-\frac{1}{2}}\theta_2, \theta_1\theta_2\right\}$ and $\left\{\theta_1 - \mathrm{i}q^{\frac{1}{2}}\theta_2\right\}$, corresponding to the spin one-half and spin zero representations. It means that the representation of the Casimir operator on $V_2 \otimes V_2$ is presented as a decomposition

$$c_{2\times 2} = \left[\frac{3}{2}\right]_q^2 P_3 + \left[\frac{1}{2}\right]_q^2 P_1 = \left[\frac{3}{2}\right]_q^2 \mathbb{1} - \left(\left[\frac{3}{2}\right]_q^2 - \left[\frac{1}{2}\right]_q^2\right) P_1, \tag{17}$$

over projection operators P_i, $P_iP_j = \delta_{ij}P_i$, defined on the states with spin zero and spin one-half. The eigenvalues of Casimir operator (17) become degenerate when $q^2 = -1$, i.e. $\left[\frac{3}{2}\right]_q^2 = \left[\frac{1}{2}\right]_q^2$. This degeneracy of the eigenvalues is not accompanied by degeneracy of the eigenvectors: vectors $\theta_1 - \mathrm{i}q^{-\frac{1}{2}}\theta_2$ and $\theta_1 - \mathrm{i}q^{\frac{1}{2}}\theta_2$ still remain linearly independent. So the rule (17) for the fusion of the couple of two-dimensional representations is valid for the exceptional values of q as well.

$\mathbf{V}_2 \otimes \mathbf{V}_3$. In this case from (16) one obtains that on the space of the tensor product $\{1, \theta_1\} \otimes \{1, \theta_2, x_2\} = \{1, \theta_2, x_2, \theta_1, \theta_1\theta_2, \theta_1 x_2\}$ the quadratic Casimir operator can be written as a sum of the projection operators

$$c_{2\times 3} = [2+\lambda]_q^2 P_4 + [1+\lambda]_q^2 P_2 = [2+\lambda]_q^2 \mathbb{1} - \left([2+\lambda]_q^2 - [1+\lambda]_q^2\right) P_2., \tag{18}$$

on the two- and four-dimensional states,

$$\begin{aligned} V_2 &= \left\{\theta_1 - \mathrm{i}q^{\frac{1}{2}}\theta_2, (q-1)x_2 - \mathrm{i}q^{-\frac{1}{2}}\theta_1\theta_2\right\}, \\ V_4 &= \left\{1, q\theta_1 - \mathrm{i}(q^{\frac{1}{2}} - q^{-\frac{1}{2}})\theta_2, x_2 + iq^{\frac{1}{2}}\theta_1\theta_2, \theta_1 x_2\right\}. \end{aligned} \tag{19}$$

The projectors P_4 and P_2 have poles at $q^3 = -1$, which means that the separation of the space spanned by vectors $\{1, \theta_1, \theta_2, \theta_1\theta_2, x_2, \theta_1 x_2\}$ into V_4 and V_2 no longer makes sense. Both the eigenvalues and eigenvectors of the Casimir operator have degeneracy at that point (i.e. $[2+\lambda]_q = -[1+\lambda]_q$ and two vectors $\{q\theta_1 - \mathrm{i}(q^{\frac{1}{2}} - q^{-\frac{1}{2}})\theta_2, x_2 + \mathrm{i}q^{\frac{1}{2}}\theta_1\theta_2\}$ belonging to the four-dimensional representation are linearly dependent with the vectors of two-dimensional representation space $\{\theta_1 - \mathrm{i}q^{\frac{1}{2}}\theta_2, (q-1)x_2 - \mathrm{i}q^{-\frac{1}{2}}\theta_1\theta_2\}$). The whole representation space contains two more vectors linearly independent with $V \equiv \{v_i\} = \left\{1, \theta_1 - \mathrm{i}q^{\frac{1}{2}}\theta_2, x_2 + \mathrm{i}q^{\frac{1}{2}}\theta_1\theta_2, \theta_1 x_2\right\}_{q^3=-1}$ vectors, and they can be chosen as $U = \{u_2, u_3\} = \{\theta_1, \theta_1\theta_2\}$. The action of the generators on the whole space is

$$\begin{aligned} e\cdot\{v_1, v_2, v_3, v_4, u_2, u_3\} &= \left\{-\mathrm{i}q^{\frac{1}{2}}v_2, -\mathrm{i}q^{\frac{1}{2}}v_3, 0, 0, u_3, -v_4\right\}, \\ f\cdot\{v_1, v_2, v_3, v_4, u_2, u_3\} &= \left\{0, 0, v_2, v_3, v_1, -\mathrm{i}q^{-\frac{1}{2}}v_2 - \mathrm{i}q^{-\frac{3}{2}}u_2\right\}, \\ k\cdot\{v_1, v_2, v_3, v_4, u_2, u_3\} &= \left\{-\mathrm{i}q^{-\frac{3}{2}}v_1, -\mathrm{i}q^{-\frac{1}{2}}v_2, -\mathrm{i}q^{\frac{1}{2}}v_3, -\mathrm{i}q^{\frac{3}{2}}v_4, -\mathrm{i}q^{-\frac{1}{2}}u_2, -\mathrm{i}q^{\frac{1}{2}}u_3\right\}. \end{aligned} \tag{20}$$

In this way, one sees that at $q^3 = -1$ the action of the algebra generators G on the tensor product $V_2 \otimes V_3$ acquires block-triangular form: $G\cdot V \Rightarrow V, G\cdot U \Rightarrow U + V$ (algebra generators G map the vectors belonging to V into themselves and map the vectors forming U into the vectors of the spaces U and V). So $V_2 \otimes V_3$ at $q^3 = -1$ has to be considered itself as an indecomposable six-dimensional representation, with proper sub-representation V_4: we denote it by $\widehat{\bar{V}_4 \oplus V_2}$ or by $\mathcal{I}^{(6)}_{\{4,2\}}$. Here the 'bar' over V_4 means that $\{v_1, v_2, v_3, v_4\}$ at $q^3 = -1$ is not irreducible and contains the invariant two-dimensional subspace $\{v_2, v_3\}$ (see (20)). $\mathcal{I}^{(6)}_{\{4,2\}}$ has two lowest and two highest weights. So, we find

$$V_2 \otimes V_3 = \begin{cases} \mathcal{I}^{(6)}_{\{4,2\}}, & \text{if} \quad q^3 = -1, \\ V_2 \oplus V_4, & \text{for other cases}, \end{cases} \tag{21}$$

Note that Casimir operator (18) remains regular at $q^3 = -1$, but it is no longer diagonal on $\{V, U\}$:

$$\left(c_{2\times 3} + \tfrac{1}{3}\mathbb{1}\right)\cdot\{v_1, v_2, v_3, v_4, u_2, u_3\} = \left\{0, 0, 0, 0, 2qv_2, 2\mathrm{i}q^{\frac{3}{2}}v_3\right\}\Big|_{q^3=-1},$$

acquiring triangular form on the vectors with weights $h = \pm\frac{1}{2} + \lambda$.

Definition. *Hereafter, by* $\mathcal{I}^{(r)}_{\{k,r-k\}}$ *(letting* $k > r-k$*) we shall denote the non-irreducible representations of dimension r, which appear in the fusions instead of the direct sum* $V_k \oplus V_{r-k}$*, when q takes exceptional values (see for detailed and general descriptions the previous section).*

$\mathbf{V}_3 \otimes \mathbf{V}_3$. Acting by finite-difference operators (2), (3) on the tensor product of two spin one-half representations $\{1, \theta_1, x_1\}$ and $\{1, \theta_2, x_2\}$, one can calculate the eigenvectors of the Casimir operator

$$V_5 = \{\varphi_5{}^{\alpha}\} \equiv \{1, \theta_1 + q\theta_2, x_1 + (q - q^2)\theta_1\theta_2 + q^2x_2, x_1\theta_2 + qx_2\theta_1, x_1x_2\},$$
$$V_3 = \{\varphi_3{}^{i}\} \equiv \{q\theta_1 - \theta_2, x_1 + (1+q)\theta_1\theta_2 - x_2, qx_1\theta_2 - x_2\theta_1\}, \qquad V_1 = \{x_1 + \theta_1\theta_2 - q^{-1}x_2\}. \tag{22}$$

Here φ are the eigenvectors of c and $\alpha = 1, 2, \ldots, 5$, $i = 1, 2, 3$. In this way, one explicitly constructs $V_3 \otimes V_3$ tensor product decomposition as the sum of representations $V_1 \oplus V_3 \oplus V_5$

$$c_{3\times 3} = \left[\frac{5}{2}\right]_q^2 P_5 + \left[\frac{3}{2}\right]_q^2 P_3 + \left[\frac{1}{2}\right]_q^2 P_1$$
$$= \left[\frac{5}{2}\right]_q^2 I + \left(\left[\frac{3}{2}\right]_q^2 - \left[\frac{5}{2}\right]_q\right) P_3 + \left(\left[\frac{1}{2}\right]_q^2 - \left[\frac{5}{2}\right]_q^2\right) P_1. \tag{23}$$

The projection operators P_1, P_3, P_5 have the following multipliers correspondingly: $(q - 1 + q^{-1})^{-1}$, $(q + q^{-1})^{-1}$ and $(q + q^{-1})^{-1}(q - 1 + q^{-1})^{-1}$. The poles of the operators P_r correspond to three different cases $q^4 = 1$, $q^6 = 1$ and $q^8 = 1$.

When $q = \pm\mathrm{i}$, all eigenvalues coincide with each other: $\left[\frac{5}{2}\right]_q^2 = \left[\frac{3}{2}\right]_q^2 = \left[\frac{1}{2}\right]_q^2$, and six eigenvectors coincide with each other pairwise (degeneracy of the eigenvectors shows itself as a linear dependence of vectors, we denote this relation between vectors as $\approx$): $\{\varphi_5{}^{\alpha+1}\} \approx \{\varphi_3{}^{\alpha}\}, \alpha = 1, 2, 3$; two of three projectors, P_5 and P_3, become singular, correspondingly the sum of the representations V_5 and V_3 transforms into one new indecomposable representation, $\mathcal{I}^{(8)}_{\{5,3\}} = \widehat{\bar{V}_5 \oplus V_3}$. The set of the vectors (22) has to be completed by three new vectors to form a basis, which can be taken as $\{1, \theta_1 + \mathrm{i}\theta_2, \theta_1 - \mathrm{i}\theta_2, x_1 + (1+\mathrm{i})\theta_1\theta_2 - x_2, x_1 + \theta_1\theta_2 + \mathrm{i}x_2, 2x_1, x_1\theta_2 + \mathrm{i}x_2\theta_1, x_1\theta_2 - \mathrm{i}x_2\theta_1, x_1x_2\}$. Consider next the case $q^3 = -1$, when $[1/2]_q^2 = 1/3 = [5/2]_q^2$. The projectors P_5 and P_1 are ill-defined at that point and one can see that $\varphi_5{}^3 \approx \varphi_1$ is the only degeneracy which occurs in this case, and $\bar{V}_5, V_1$ are unified into a six-dimensional indecomposable representation $\mathcal{I}^{(6)}_{\{5,1\}}$. Finally, the last cases to be considered ($q^4 = -1, q^3 = 1$) are not degenerate, all the vectors (22) are distinct from each other and all the projection operators P_5, P_3 and P_1 are well defined at these points,

$$V_3 \otimes V_3 = \begin{cases} V_1 \oplus \mathcal{I}^{(8)}_{\{5,3\}}, & q^2 = -1, \\ \mathcal{I}^{(6)}_{\{5,1\}} \oplus V_3, & q^3 = -1, \\ V_1 \oplus V_3 \oplus V_5, & \text{otherwise}. \end{cases} \tag{24}$$

$\mathbf{V}_3 \otimes \mathbf{V}_5$. Fifteen vectors which form basis of the tensor product $(\{1, \theta_1, x_1\} \otimes \{1, \theta_2, x_2, x_2\theta_2, x_2^2\})$ are decomposed into the direct sum $V_3 \otimes V_5 = V_3 \oplus V_5 \oplus V_7$ for the general values of q,

$$c_{3\times 5} = \left[\frac{3}{2}\right]_q^2 P_3 + \left[\frac{5}{2}\right]_q^2 P_5 + \left[\frac{7}{2}\right]_q^2 P_7. \tag{25}$$

Analysis shows that the following degeneracies take place: $c_7 = c_5, \{\varphi_7^{a+1}\} \approx \{\varphi_5^{a}\}$ at $q^3 = \pm 1$, $c_7 = c_3$, $\{\varphi_7^{i+2}\} \approx \{\varphi_3^{i}\}$ at $q^5 = -1$ and $c_5 = c_3$, $\{\varphi_5^{i+1}\} \approx \{\varphi_3^{i}\}$ at $q^2 = -1$.

For these exceptional cases the conventional spins addition rule (25) does not work: some homogeneous vectors belonging to different items on the rhs of (25) coincide with each other. The fusion rules at any value of q look like:

$$V_3 \otimes V_5 = \begin{cases} V_3 \oplus \mathcal{I}^{(12)}_{\{7,5\}}, & q^3 = 1, \\ \mathcal{I}^{(10)}_{\{7,3\}} \oplus V_5, & q^5 = -1, \\ V_3 \otimes V_5 \otimes V_7, & q^3 \neq \pm 1, \quad q^5 \neq -1, \quad q^2 \neq -1. \end{cases} \tag{26}$$

As we saw in (24) the representation V_5 is absent among the items in the decomposition of the tensor product $V_3 \otimes V_3$ for the values $q^2 = -1, q^3 = -1$. To explain this fact let us consider what happens with V_5 in the limits $q^2 = -1$ or $q^3 = -1$. We find that this five-dimensional representation is non-completely reducible (it has a proper subspace which remains invariant under action of algebra generators). We denote such representations by $\bar{V}_d$, as we did in the previous examples for the maximal proper sub-representations of $\mathcal{I}^{(r)}_{\{k,r-k\}}$-representations. The fusion rules can be written down for such representations as well,

$$V_3 \otimes \bar{V}_5 = \begin{cases} \bar{V}_7 \oplus \mathcal{I}^{(8)}_{\{5,3\}}, & q^2 = -1, \\ V_3 \oplus \mathcal{I}^{(12)}_{\{7,5\}}, & q^3 = -1. \end{cases} \tag{27}$$

Note that representations $\mathcal{I}^{(12)}_{\{7,5\}}$ for the cases $q^3 = -1$ and $q^3 = 1$ are different: as we already know, the representation V_5 is an irrep when $q^3 = 1$, in contrast to the first case. And at $q^3 = -1$ the representation $\mathcal{I}^{(12)}_{\{7,5\}}$ has more than two lowest and two highest weights.

As was mentioned (see also [12]), the matrix representing the Casimir operator c for the indecomposable representations besides diagonal part also contains non-diagonalizable triangular blocks, which couple the eigenvectors with same k-values.

To finish this section we would like to consider one more example:

$\mathbf{V}_3 \otimes \mathbf{V}_3 \otimes \mathbf{V}_3$. For general values of q using the associativity of the tensor product we can write

$$\begin{aligned} V_3 \otimes V_3 \otimes V_3 &= V_3 \otimes (V_1 \oplus V_3 \oplus V_5) = V_3 \oplus (V_3 \otimes V_3) \oplus (V_3 \otimes V_5) \\ &= V_1 \oplus 3(V_3) \oplus 2(V_5) \oplus V_7. \end{aligned} \tag{28}$$

Here multipliers 2 and 3 mean multiplicities of the corresponding items in the decomposition. One can calculate the eigenvectors of the Casimir operator:

$$\begin{aligned} &c \cdot \varphi_1 = [1/2]_q^2 \varphi_1, && c \cdot \varphi_7^\alpha = [7/2]_q^2 \varphi_7^\alpha, && \alpha = 1, 2, \ldots 7, \\ &c \cdot \varphi_5^i = [5/2]_q^2 \varphi_5^i, && c \cdot \varphi'^i_5 = [5/2]_q^2 \varphi'^i_5, && i = 1, 2, 3, 4, 5, \\ &c \cdot \left\{\varphi_3^a, \varphi'^a_3, \varphi''^a_3\right\} = [3/2]_q^2 \left\{\varphi_3^a, \varphi'^a_3, \varphi''^a_3\right\}, && && a = 1, 2, 3, \end{aligned} \tag{29}$$

where $\varphi_r^a, \varphi'^a_r, \varphi''^a_r$ are the eigenvectors corresponding to the r-dimensional representations in the decomposition above. Taking into account the previous analysis, one expects that the vectors φ become linearly dependent for the same values of q as for $(V_3 \otimes V_3)$ and $(V_3 \otimes V_5)$. For the cases $q^3 = 1, q^5 = -1$, when spin-1 irrep V_5 exists, from (28) and (26) it follows that

$$V_3 \otimes V_3 \otimes V_3 = \begin{cases} V_1 \oplus 3(V_3) \oplus V_5 \oplus \mathcal{I}^{(12)}_{\{7,5\}}, & q^3 = 1 \\ V_1 \oplus 2(V_3) \oplus 2(V_5) \oplus \mathcal{I}^{(10)}_{\{7,3\}}, & q^5 = -1. \end{cases} \tag{30}$$

This rule can be obtained also from the direct analysis of the eigenvectors (29). The following degeneracies take place: $q^2 \varphi_7^{i+1} - \varphi_5^i + q^{-2} \varphi'^i_5 = 0$, when $q^3 = 1, i = 1, \ldots, 5$ $(q^3 = -1, i \neq 3)$ the spin-3/2 and a combination of two spin-1 representations are unified into one representation: $\mathcal{I}^{(12)}_{\{7,5\}} = \widehat{\bar{V}_7 \oplus V_5}$, at $q^3 = \pm 1$, and an orthogonal combination is unified with φ_0 into $\mathcal{I}^{(6)}_{\{5,1\}} = \widehat{\bar{V}_5 \oplus V_1}$ at $q^3 = -1$.

For $q^5 = -1$ the following relation takes place: $\varphi_7^{a+2} \approx \left(q^{-1}\varphi_3^a + \varphi^{a'}{}_3 + q\varphi^{a''}{}_3\right)$, which means that the sum of spin-3/2 and a combination of three spin-1/2 irreps transforms into $\mathcal{I}^{(10)}_{\{7,3\}} = \widehat{\bar{V}_7 \oplus V_3}$.

Finally, when $q^2 = -1$, we have $\varphi_7^4 + q\varphi_1 = 0$ and

$$\varphi_5^{a+1} - (1+q)\varphi_3^a + \varphi^{a'}{}_3 + q\varphi^{a''}{}_3 = 0, \qquad \varphi'^{a+1}_5 + \varphi_3^a - q^{-1}\varphi^{a'}{}_3 - (1+q)\varphi^{a''}{}_3 = 0,$$

these degeneracies indicate that when $q^2 = -1$ the vectors of two spin-1 irreps are unified with two combinations of three spin-1/2 irreps and produce two $\mathcal{I}^{(8)}_{\{5,3\}} = \widehat{\bar{V}_5 \oplus V_3}$-representations, and another indecomposable representation $\mathcal{I}^{(8)}_{\{7,1\}}$ arises from the unification of spin-3/2 and spin-0 irreps: $\widehat{\bar{V}_7 \oplus V_1}$. More correctly, at the exceptional values of q the corresponding items in the tensor product decomposition (29) are replaced by indecomposable representations.

With respect to the case (26) new features appear only for the values $q^2 = -1$, $q^3 = -1$,

$$V_3 \otimes V_3 \otimes V_3 = \begin{cases} 2\left(\mathcal{I}^{8}_{\{5,3\}}\right) \oplus \mathcal{I}^{(8)}_{\{7,1\}} \oplus V_3, & q^2 = -1, \\ \mathcal{I}^{(12)}_{\{7,5\}} \oplus 3(V_3) \oplus \mathcal{I}^{(6)}_{\{5,1\}}, & q^3 = -1. \end{cases} \tag{31}$$

They provide us with decomposition rules for the tensor products $V_3 \otimes \mathcal{I}^{(8)}_{\{5,3\}}$ and $V_3 \otimes \mathcal{I}^{(6)}_{\{5,1\}}$. Using the associativity property of the tensor product one can deduce

$$\begin{aligned} &\mathcal{I}^{(8)}_{\{5,3\}} \otimes V_3 = 2\left(\mathcal{I}^{(8)}_{\{5,3\}}\right) \oplus \mathcal{I}^{(8)}_{\{7,1\}}, && q^2 = -1, \\ &\mathcal{I}^{(6)}_{\{5,1\}} \otimes V_3 = \mathcal{I}^{(12)}_{\{7,5\}} \oplus 2(V_3), && q^3 = -1. \end{aligned} \tag{32}$$

Here the representation $\mathcal{I}^{(12)}_{\{7,5\}}$ is decomposable into two six-dimensional indecomposable representations (with the eigenvalues of k respectively $\{q^{\frac{3}{2}}, q, q, q^{\frac{1}{2}}, q^{\frac{1}{2}}, 1\}$ and $\{1, q^{-\frac{1}{2}}, q^{-\frac{1}{2}}, q^{-1}, q^{-1}, q^{-\frac{3}{2}}\}$), which do not coincide with $\mathcal{I}^{(6)}_{\{5,1\}}$, rather having the structure of $\mathcal{I}^{(6)}_{\{4,2\}}$ (20)!

We note that there is a correspondence between the above decompositions (32) and the results in (27), which is the consequence of $\bar{V}_5 \subset \mathcal{I}^{(8)}_{\{5,3\}}, \mathcal{I}^{(6)}_{\{5,1\}}$ and $\bar{V}_7 \subset \mathcal{I}^{(8)}_{\{7,1\}}, \mathcal{I}^{(12)}_{\{7,5\}}$. The only difference is that $\mathcal{I}^{(12)}_{\{7,5\}}$ ($q^3 = -1$) is decomposable in (32): this difference arises from the distinction in the structure of the representation $\bar{V}_5$, as for the first case we have fixed it by direct choosing the representation's polynomial space as $\{1 = f^4 \cdot x^2/[2]_q, \ldots, f \cdot x^2/[2]_q, x^2\}$, while in the second case $\bar{V}_5$ emerges from the product $V_3 \otimes V_3$ and here $\Delta(f)^3 = 0(\Delta(e)^3 = 0)$ at $q^3 = -1$.

One can also check directly the following fusion for representation $\mathcal{I}^{(8)}_{\{7,1\}}$, appeared in (32),

$$\mathcal{I}^{(8)}_{\{7,1\}} \otimes V_3 = \mathcal{I}^{(16)}_{\{9,7\}} \oplus \mathcal{I}^{(8)}_{\{5,3\}}, \qquad q^2 = -1, \tag{33}$$

where $\mathcal{I}^{(16)}_{\{9,7\}}$ representation arises from the merging of V_9 and V_7. This representation consists of two indecomposable representations in the form of $\mathcal{I}^{(8)}_{\{5,3\}}$, with the eigenvalues of k being respectively $\left\{q^2, q^{\frac{3}{2}}, q^{\frac{3}{2}}, q, q, q^{\frac{1}{2}}, q^{\frac{1}{2}}, 1\right\}$ and $\left\{1, q^{\frac{-1}{2}}, q^{\frac{-1}{2}}, q^{-1}, q^{-1}, q^{\frac{-3}{2}}, q^{\frac{-3}{2}}, q^{-2}\right\}$.

We have seen in the considered examples that non-irreducible representation $\mathcal{I}^{(r)}_{\{k,r-k\}}$, emerging from the *multiple tensor products* of the irreps, is an indecomposable representation only in the case, when V_{r-k} is an irrep for the given q.

Then quartic product of the irreps V_3 gives decomposition rule for $\mathcal{I}^{(8)}_{\{5,3\}} \otimes \mathcal{I}^{(8)}_{\{5,3\}}$ when $q^2 = -1$:

$$\begin{aligned}
\otimes^4 V_3 &= \otimes^2\big(\mathcal{I}^{(8)}_{\{5,3\}} \oplus V_1\big) = \big(\mathcal{I}^{(8)}_{\{5,3\}} \otimes \mathcal{I}^{(8)}_{\{5,3\}}\big) \oplus 2\big(\mathcal{I}^{(8)}_{\{5,3\}}\big) \oplus V_1 \\
&= \big(2\big(\mathcal{I}^{(8)}_{\{5,3\}}\big) \oplus \mathcal{I}^{(8)}_{\{7,1\}} \oplus V_3\big) \otimes V_3 = 6\big(\mathcal{I}^{(8)}_{\{5,3\}}\big) \oplus 2\big(\mathcal{I}^{(8)}_{\{7,1\}}\big) \oplus \big(\mathcal{I}^{(16)}_{\{9,7\}}\big) \oplus V_1, \\
&\Rightarrow \mathcal{I}^{(8)}_{\{5,3\}} \otimes \mathcal{I}^{(8)}_{\{5,3\}} = 4\big(\mathcal{I}^{(8)}_{\{5,3\}}\big) \oplus 2\big(\mathcal{I}^{(8)}_{\{7,1\}}\big) \oplus \big(\mathcal{I}^{(16)}_{\{9,7\}}\big) = 6\big(\mathcal{I}^{(8)}_{\{5,3\}}\big) \oplus 2\big(\mathcal{I}^{(8)}_{\{7,1\}}\big).
\end{aligned} \tag{34}$$

Moreover for the case $q^2 = -1$ one can sketch out the fusion of the tensor product of spin-1/2 representations of number k in closed form, as follows (see details in section 5):

$$\underbrace{V_3 \otimes V_3 \cdots \otimes V_3}_{k} = \begin{cases} \bigoplus_{p=1;\alpha}^{k/2} \varepsilon_k(p,\alpha)\mathcal{I}^{(8p)}_{\alpha} \oplus V_1, & \text{for even} \quad k, \\ \bigoplus_{p=1;\alpha}^{(k-1)/2} \varepsilon_k(p,\alpha)\mathcal{I}^{(8p)}_{\alpha} \oplus V_3, & \text{for odd} \quad k, \end{cases} \tag{35}$$

here $\varepsilon(p,\alpha)$ stands for multiplicity of $\mathcal{I}^{(8p)}_{\alpha}$, where $\alpha = \{4p+1, 4p-1\}, \{4p+3, 4p-3\}$.

4. Tensor product of arbitrary lowest weight irreps and projectors

The fusion of two arbitrary irreps $V_n \otimes V_m$. The even- and odd-dimensional irreps can be considered on an equal footing. Suppose we have two finite-dimensional lowest weight representations V_n and V_m at general values of q and let for definiteness $n \leqslant m$. Then eigenvalues of the Casimir operator are built as follows: the lowest weight vectors φ_i are defined as solutions to the equation

$$f \cdot \varphi_i = (f_1 + k_1 f_2)\varphi_i = 0. \tag{36}$$

The number of these solutions is precisely equal to n because they are built using n independent vectors of V_n. Then each lowest weight vector gives rise to the invariant subspace of the Casimir operator by successive action of the rising operator e (8) on that vector. The invariant subspace with largest dimension contains the lowest weight vector 1 and the highest one with weight $\frac{n-1}{4} + \frac{m-1}{4}$, i.e. has dimension $n + m - 1$. It means that decomposition (16) contains exactly n terms: $V_n \otimes V_m = V_{m-n+1} \oplus V_{m-n+3} \oplus \cdots \oplus V_{m+n-1}$. Consequently, the Casimir operator has the following decomposition over the corresponding projection operators (in accordance with (16)):

$$c_{n\times m} = c_{m-n+1}P_{m-n+1} + c_{m-n+3}P_{m-n+3} + \cdots + c_{m+n-1}P_{m+n-1}, \tag{37}$$

where c_r is the eigenvalue of the Casimir operator on r-dimensional invariant subspace:

$$c_r = (-1)^{r+1}\frac{q^r + (-1)^r 2 + q^{-r}}{q^2 - 2 + q^{-2}} = \begin{cases} \left[\dfrac{r}{2}\right]_q^2, & \text{if} \quad r \text{ is odd}, \\ \left[\dfrac{r}{2} + \lambda\right]_q^2, & \text{if} \quad r \text{ is even}. \end{cases} \tag{38}$$

In expression (37) the projector P_r, defined on the invariant subspace with given eigenvalue c_n of the Casimir operator $c_{n\times m}$, can be written as

$$P_r = \prod_{p\neq r} \frac{c_{n\times m} - c_p \mathbb{1}}{c_r - c_p}, \qquad \sum_r P_r = \mathbb{1}, \qquad P_r P_p = \delta_{rp} P_r. \tag{39}$$

The Casimir eigenvalues c_r (38) coincide with each other only at exceptional values of q: $c_{r_1} = c_{r_2}$ is equivalent to the equation

$$((-q)^{r_1+r_2} - 1)((-q)^{r_1-r_2} - 1) = 0. \tag{40}$$

Some projectors then become ill-defined, having singularities, which is a sign that when equation (40) occurs, then the decomposition (37) is no longer valid, and the spaces with the ill-defined projectors can be unified into indecomposable representations. However this condition is necessary but not sufficient: zeros in the numerator and denominator in (39) can cancel each other and some projection operators can survive as happened in the examples considered above: for the tensor product of two two-dimensional representations Casimir operator at the special points $q = \pm\mathrm{i}$ turns to be a multiple of the unity matrix: $c = \frac{1}{2}\mathbb{1}$, and both projectors $P_1 = (c - c_3\mathbb{1})/(c_1 - c_3)$, $P_3 = (c - c_1\mathbb{1})/(c_3 - c_1)$ remain regular. A similar situation occurs in the example $V_3 \otimes V_5$, when $c_3 = c_5$ at $q^4 = -1$. A careful analysis shows that at $q^4 = -1$ the Casimir operator c satisfies the relation: $(c - c_7\mathbb{1})(c - c_5\mathbb{1})|_{q^4=-1} = 0$, and the projectors P_3, P_5 survive.

Let us consider $V_n \otimes V_m$ for the cases with $n = 2, 3$ separately.

The tensor product of two-dimensional and an arbitrary irrep. The fusion rule is

$$V_2 \otimes V_m = V_{m-1} \oplus V_{m+1}, \tag{41}$$

and only two projection operators exist: $c_{2\times m} = c_{m-1}P_{m-1} + c_{m+1}P_{m+1}$,

$$P_{m-1} = \frac{1}{c_{m+1} - c_{m-1}}(c_{m+1}\mathbb{1} - c_{2\times m}), \qquad P_{m+1} = \frac{1}{c_{m+1} - c_{m-1}}(-c_{m-1}\mathbb{1} + c_{2\times m}). \tag{42}$$

From this form of projection operators one immediately deduces that the only indecomposable representation which can appear in the case of $c_{m+1} = c_{m-1}$ is $\mathcal{I}^{(2m)}_{\{m+1,m-1\}}$, when $q^{2m} = 1$ (40).

The tensor product of three- and an arbitrary-dimensional irrep. In this case for general values of q the following decomposition is valid:

$$V_3 \otimes V_r = V_{r-2} \oplus V_r \oplus V_{r+2}$$

and one has $c_{3\times r} = c_{r+2}P_{r+2} + c_rP_r + c_{r-2}P_{r-2}$, where the structure of the denominators in the expressions of P_r (see (39)) suggests that the projectors can be singular when $c_r = c_{r-2}$, $c_{r+2} = c_{r-2}$ and/or $c_{r+2} = c_r$. As we shall see, only two kinds of indecomposable representations can appear in this fusion: $\mathcal{I}^{2r}_{\{r+2,r-2\}}$ $(c_{r+2} = c_{r-2}, q^{2r} = 1, q^4 = 1)$ and $\mathcal{I}^{2r+2}_{\{r+2,r\}}(c_{r+2} = c_r, q^{2r+2} = 1)$.

5. General results: fusion rules

The aim of this section is to clarify the peculiarities of the finite-dimensional representations and their fusions which occur at exceptional values of $q, q^N = \pm 1$, for general $N \in \mathbb{N}$. The considered examples show that the number of irreducible representations is restricted, when q is given by a root of unity, and the new type of representations—indecomposable representations—appears in the fusions. Is it possible to extend observed regularities to general $N \in \mathbb{N}$, finding all finite-dimensional non-reducible representations with their fusion rules, and the relations between N and the dimensions of the permissible representations?

As we have already seen above, when $q^N = \pm 1$, the irrep V_r, since $r > r_{\max}$, becomes non-irreducible representation $\bar{V}_r$, which contains one or more proper subspaces. Such representations do not appear in the fusions of the irreps, but indecomposable representations appearing in the tensor products' decompositions contain such representations as sub-representations ($\mathcal{I} = \widehat{\bar{V} \oplus V}$). This observation allows us to trace the connection between the number N and the dimensions of the permissible irreps (i.e. $r_{\max}$) and the indecomposable representations.

All the representations can be constructed uniformly, in a general form. As usual, one can choose as basis vectors of a representation the eigenvectors $|h_n\rangle$ of the operator k,

$$k|h_n\rangle = k_n|h_n\rangle, \qquad k_n = q^{h_n} \in \mathbb{C}. \tag{43}$$

Then from the algebra relations (1) one obtains constraints on the actions of the operators e and f:

$$k(e^m|h_n\rangle) = q^m k_n(e^m|h_n\rangle), \qquad k(f^m|h_n\rangle) = q^{-m}k_n(f^m|h_n\rangle).$$

If $e^m|h_n\rangle \neq 0$ and $f^m|h_n\rangle \neq 0$, then $e^m|h_n\rangle \approx |h_n+m\rangle$ and $f^m|h_n\rangle \approx |h_n-m\rangle$ are also the eigenvectors of k-operator, with the eigenvalues of k being the powers of q, $q^{h_n\pm m}$. But for $q^N = 1$ the spectrum of the eigenvalues of the operator k becomes degenerate: the states $|h_n\rangle, |h_n \pm N\rangle, |h_n \pm 2N\rangle, \ldots$ have the same eigenvalue of k. It means that in this case one has

$$f|h_n\rangle = \alpha_1|h_n-1\rangle + \alpha_2|h_n-1\pm N\rangle + \cdots, e|h_n\rangle = \alpha'_1|h_n+1\rangle + \alpha'_2|h_n+1\pm N\rangle + \cdots. \tag{44}$$

The parameters α_i, α'_i define the representation, and the anti-commutation relation between e and f imposes constraints on them. Different values of these parameters correspond to reducible or non-reducible representations (cyclic, semi-cyclic, nilpotent or lowest/highest weight ones [14]). In particular, at general values of q the finite-dimensional irreps can be found suggesting the existence of the lowest weight vector $|h_0\rangle$, $f|h_0\rangle = 0$. Then the following relation takes place:

$$fe^r|h_0\rangle = ([h_0+r-1]_q - [h_0+r-2]_q + \cdots + (-1)^{r-1}[h_0]_q)e^{r-1}|h_0\rangle. \tag{45}$$

If the rhs vanishes, then the representation $\{|h_0\rangle, |h_1\rangle = f|h_0\rangle, \cdots |h_r\rangle = e^{r-1}|h_0\rangle\}$ is an r-dimensional irreducible lowest weight (by construction) representation, and the possible values of h_0 can be obtained from the analysis of the zeros of (45), which gives $q^{2h_0} = (-1)^{r-1}q^{1-r}$. For odd values of r, the eigenvalues h_p take integer values ($h_p \in \{(1-r)/2, (3-r)/2, \ldots, (r-1)/2\}$, q-analog of the conventional spin irreps with spin $(r-1)/4$), while for even-dimensional irreps the values h_0 contain the nontrivial term $(\imath\pi/(2\log q))$ [15], see (14). For the exceptional values of q, as was already mentioned in section 2, the lowest/highest weight representations emerged from the fusions of the fundamental spin-half irreps are distinguished by the values equal to 0 of the operators $e^{\mathcal{N}}, f^{\mathcal{N}}$, where $\mathcal{N} = \left\{\begin{smallmatrix} N, \text{even } N \\ 2N, \text{odd } N\end{smallmatrix}\right.$ if $q^N = 1$ or $\mathcal{N} = \left\{\begin{smallmatrix} N, \text{odd } N \\ 2N, \text{even } N\end{smallmatrix}\right.$ if $q^N = -1$.

5.1. Odd-dimensional conventional representations and indecomposable representations

The odd-dimensional representations for general values of q form a closed fusion (16)

$$V_{4j_1+1} \otimes V_{4j_2+1} = \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} V_{4j+1}, \qquad \Delta j = \frac{1}{2}, \tag{46}$$

where j is integer or half-integer. In this part, we consider only odd-dimensional representations and their fusions at roots of unity, but the whole analysis can be carried out with the inclusion of the even-dimensional ones as well. It is presented in the following section.

Representation V. For the general values of q, the action of the generators e, f and k on the vectors $V_{4j+1} = \{v_j(h)\}$ of the *odd-dimensional spin-j representation* can be written as

$$\begin{cases} f\cdot v_j(h) = \gamma_h^j(q)v_j(h-1), & -2j < h \leqslant 2j, \quad f\cdot v_j(-2j) = 0, \\ k\cdot v_j(h) = q^h v_j(h), & -2j \leqslant h \leqslant 2j, \\ e\cdot v_j(h) = \beta_h^j(q)v_j(h+1), & -2j \leqslant h < 2j, \quad e\cdot v_j(2j) = 0, \end{cases}$$
$$\gamma_h^j(q)\beta_{h-1}^j(q) = \alpha_h^j(q). \tag{47}$$

The algebra relations imply the following expressions for the coefficients $\alpha_h^j(q)$:

$$\alpha_h^j(q) = \sum_{i=h}^{2j}(-1)^{i-h}[i]_q = \frac{(-1)^{2j+h}[2j+1/2]_q + [h-1/2]_q}{\sqrt{q}+1/\sqrt{q}}, \qquad -2j < h \leqslant 2j, \quad (48)$$

$$\alpha_h^j(q) = -\alpha_{-h+1}^j(q), \qquad -2j < h \leqslant 0. \quad (49)$$

Usually the coefficients $\beta_h^j(q), \gamma_h^j(q)$ are chosen imposing some normalization conditions on the basis vectors (e.g. defining a norm [4], such that $\langle v_j(h)|v_j(h)\rangle = 1$, if the lowest weight vector $v_j(-2j)$ has parity 0, and $\langle v_j(h)|v_j(h)\rangle = 1(-1)$ for $2j-h$ even integer (odd integer), if the lowest weight vector has parity 1; and setting f to be the adjoint of e, when the adjoint of an operator g is defined as $\langle g^*\cdot v|u\rangle = (-1)^{p(v)p(u)}\langle v|g\cdot u\rangle$). But it fails when q is given by a root of unity.

Here $\beta_h^j(q), \gamma_h^j(q)$ define the sub-structure of the representation. However the possible choice does not affect the conclusions given below for the fusion rules. One can formulate the following.

Statement I. The representation V_{4j+1} contains invariant sub-representations, if at least one of the functions $\alpha_h^j(q)$, $-2j < h \leqslant 2j$, describing V_{4j+1} is equal to zero.

If $\alpha_h^j(q) = 0$, then we call the representation V_{4j+1} non-exactly-reducible and denote it by $\bar{V}_{4j+1}$. This representation is not irrep and contains more than one highest and more than one lowest weight vectors (which can be $v_j(\pm(h-1))$).

If the functions $\beta_h^j(q), \gamma_h(q)$ in the definition (47) are chosen as

$$\begin{aligned} &\beta_{h-1}^j(q) = 1, && \gamma_h^j(q) = \alpha_h^j(q), && -2j < h < 1, \\ &\beta_{h-1}^j(q) = \alpha_h^j(q), && \gamma_h^j(q) = 1, && 1 \leqslant h \leqslant 2j, \end{aligned} \quad (50)$$

then $\alpha_{2j'+1}^j(q) = 0$ ($j' > 0$) indicates the appearance of the invariant sub-representation $\{v_j(h)\}, -2j' \leqslant h \leqslant 2j'$ inside of $\bar{V}_{4j+1}$. In figure 1(*b*), we described representation $\bar{V}_{4j_2+1} \supset V_{4j_1+1}$ diagrammatically, denoting states $v_j(h)$ by dots (the corresponding values of h are noted in the left column). In the diagram the arrows $\uparrow$ and $\downarrow$ correspond to the action of the raising and lowering operators. In figure 1, all the dots that are not shown in the diagrams are connected with their nearest neighbors with both arrows ($\uparrow$ and $\downarrow$). In this case, there are two highest weight vectors, $v_{j_2}(2j_2), v_{j_2}(2j_1)$, and two lowest weight vectors, $v_{j_2}(-2j_2), v_{j_2}(-2j_1)$. For the cases when $\bar{V}_r$ has more than two highest and two lowest weight vectors, we should depict the diagram for $\bar{V}_r$ in a similar way, omitting the $\uparrow$-arrows, connected the dots describing the highest weight vectors with their upper nearest neighbors, and $\downarrow$-arrows, connected the dots of the lowest weight vectors with their lower nearest neighbors.

Note that if one chooses $\beta_h^j(q), \gamma_h^j(q)$ to be proportional to $\sqrt{\alpha_{h+1}^j(q)}, \sqrt{\alpha_h^j(q)}$ (imposing $e = f^\tau$), then representation $\bar{V}_{4j+1}$ will be completely reducible when some $\alpha_h^j(q) = 0$, $-2j < h \leqslant 2j$ (e.g. in the example described in figure 1(*b*), $\bar{V}_{4j_2+1}$ would be split into one $(4j_1+1)$- and two $2(j_2-j_1)$-dimensional representations).

It follows from (48) that $\alpha_h^j(q) = 0$ is equivalent to the equation

$$(1-(-1)^{2j+h}q^{2j+h})(1+(-1)^{2j-h}q^{2j+1-h}) = 0. \quad (51)$$

Taking into account that $\alpha_h^j(q) = -\alpha_{-h+1}^j(q)$, we can consider only the solutions to the equation $q^{2j+h} = (-1)^{2j+h}$, which, for the whole range of the eigenvalues h, $-2j < h \leqslant 2j$,

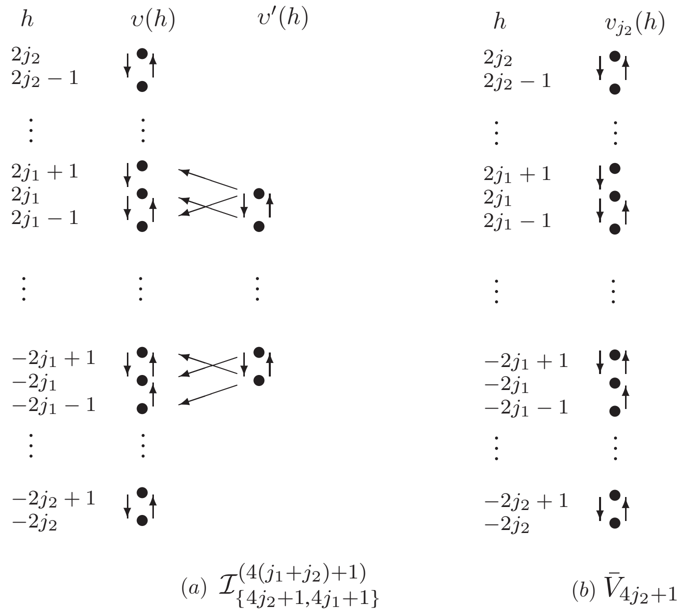

Figure 1. Representations (a) $\mathcal{I}^{(4(j_1+j_2)+1)}_{\{4j_1+1,4j_2+1\}}$ and (b) $\bar{V}_{4j_2+1} \supset V_{4j_1+1}$.

can take place if $q^n = 1$, with n being an even integer, or if $q^n = -1$, with n being an odd, and at the same time $2j + h = np$, p is positive integer, with the range in the interval $1 < np \leqslant 4j$.

We can summarize as follows: $\alpha^j_h(q) = 0$, if

$$\begin{cases} q^N = \pm 1, & N \text{ is even,} \\ q^N = 1, & N \text{ is odd,} \end{cases} \qquad \text{with} \quad h = 2Np - 2j, \tag{52}$$

$$q^{2N-1} = -1, \qquad \text{with} \quad h = (2N-1)p - 2j. \tag{53}$$

Here $1 < Np \leqslant 2j$, and the case, when $q^N = 1$ and N is even, could be omitted, as it is equivalent to the case $q^{N/2} = -1$.

So, if q satisfies one of relations (52) and (53), then for the corresponding $\{j, h\}$-s one has $\alpha^j_h(q) = 0$ and V_{4j+1} is no longer irreducible and should be denoted as $\bar{V}_{4j+1}$.

On the other hand, from (52) and (53) it follows that for a given N, the *permissible* irreps are the representations V_{4j+1} with spin j, which satisy the inequality

$$j \leqslant j_{\max}, \qquad j_{\max} = \frac{N-1}{2} \qquad \text{for} \qquad \begin{cases} q^N = 1, & N \text{ is odd} \\ q^N = -1, & N \text{ is even} \end{cases} \tag{54}$$

$$j_{\max} = \frac{N-1}{4} \qquad \text{for} \qquad q^N = -1, \quad N \text{ is odd.} \tag{55}$$

As we have already seen, $\bar{V}_{4j+1}$-representations, $j > j_{\max}$, do not emerge in the fusions of the irreps; instead new indecomposable representations appear. Let us summarize the observed regularities as follows:

Statement II. The following three criteria describe the appearance of an indecomposable representation: when in the rhs of the decomposition (46) any two representations

$$V_{4j+1} = \{v_j(-2j), \ldots, v_j(2j)\}, \qquad V_{4s+1} = \{v_s(-2s), \ldots, v_s(2s)\}, \qquad s < j,$$

(1) have the same eigenvalues of the Casimir operator, $c_{4s+1} = c_{4j+1}$—necessary criterion;
(2) the following eigenvectors of the Casimir operator are linearly dependent: $v_s(h) \approx v_j(h), h \in (-2s, \ldots, 2s)$—a necessary and sufficient criterion;
(3) $\bar{V}_{4j+1} \supset V_{4s+1}$ (i.e. V_{4j+1} turns into $\bar{V}_{4j+1}$ one), and $\bar{V}_{4j+1}$ has no larger proper sub-space than V_{4s+1}—a necessary and sufficient criterion.

It means that the sum $V_{4s+1} \oplus V_{4j+1}$ degenerates and after completion by new vectors $v'(h)$, with the eigenvalues of generator k being q^h, $h \in (-2s, \ldots, 2s)$, turns into the indecomposable representation $\mathcal{I}^{(4(s+j)+2)}_{\{4j+1,4s+1\}} = \widehat{\bar{V}_{4j+1} \oplus V_{4s+1}}$.

(1) It is easy to see that when the first point does not occur, then all the spins j in (46) are 'permissible' (54) and (55) and hence the decomposition (46) remains unchanged. But it is possible to have a situation when all the spins are 'permissible' but a casual degeneration of the eigenvalues of the Casimir operator occurs. So, the first point is the simplest necessary, but not sufficient, criterion for the distortion of the usual decomposition rule.
(2) The realization of the second point means that the mentioned vectors belonging to different representations coincide with each other, so the usual decomposition rule (46) is spoiled. Moreover, such coincidence of the Casimir eigenvectors from different multiplets immediately implies coincidence of the corresponding eigenvalues, i.e the first point follows from the second one. As V_{4s+1} is an irrep, it means that the vectors $\{v_j(-2s), \ldots, v_j(2s)\}$ constitute a proper sub-space of the representation V_{4j+1} (so, the third point is realized as well), and consequently $j > j_{\max}$.
(3) The third point implies that the equation $\alpha^j_{2s+1}(q) = 0$ takes place, and hence $(-q)^{2j+2s+1} = 1$ (statement I and (51)), i.e. $j > j_{\max}$. Note that the solutions to equations (51) are also the solutions to (40), when $r_1 = 4j+1, r_2 = 2h-1$, so if for some exceptional q the third point of statement II occurs, the first point is also true. The relation $\alpha^j_h(q) = \alpha^s_h(q)$ is fulfilled as well, when $(-q)^{2j+2s+1} = 1$. Hence the $(4s+1)$-dimensional sub-representation of V_{4j+1} and the representation V_{4s+1} have the same characteristics. It is easy to verify that any linear superposition of the vectors $v_j(h)$ and $v_s(h)$ with the weights $h \in (-2s, \ldots, 2s)$ belongs (up to numerical coefficients) either to the representation V_{4j+1} or to V_{4s+1}, which indicates that the mentioned vectors are linearly dependent (i.e. the second point follows from the third one too).

Vice versa, any destruction of the Clebsh–Gordan decomposition at roots of unity means that there must be a spin j in (46) which is larger than $j_{\max}$. Then for such representation V_{4j+1} the relation $\alpha^j_{h'+1}(q) = 0$ takes place (statement I) for some h', and consequently $(-q)^{2j+h'+1} = 1$ according to (52) and (53). And as now $\Delta(e^{2j+h'+1}) = 0$, $\Delta(f^{2j+h'+1}) = 0$ (recalling definition of $\mathcal{N}$ and formulae (10), (52) and (53)), so $\beta^j_{-h'} = \gamma^j_{h'} = 0$, which means that $\bar{V}_{4j+1}$ has $(2h'+1)$-dimensional proper sub-representation. This brings us to the situation described in the third point of statement II, with $h' \equiv 2s$, i.e. any distortion of the standard fusion rules leads to fulfillment of the third point, and consequently to the first and second points as well.

Let us now see that the coincidence of the eigenvectors leads to the appearance of the indecomposable representation. Indeed, as we know, in the decomposition (46) at general q the eigenvectors of the Casimir operator $v_j(h)$ (in rhs of the equation) make a basis in the space of the tensor product (lhs of the equation), formed by $v_{j_1}(h_1) \otimes v_{j_2}(h_2)$. The second point shows that the number of non-zero eigenvectors is reduced (some eigenvectors are identical to others). Hence it is necessary to supplement them with new vectors $v'(h)$ to span the whole space of the decomposition. In order to find the vectors $v'(h)$ we can borrow the concept of the vectors with null norm from [3] (see also references therein), where it was observed that when $v_j(\pm 2s)$ $(j > s)$ are highest and lowest weight vectors, then all the states $v_j(h), h \in (-2s, \ldots, 2s)$ have null norms. As was mentioned already, a norm can be defined in the graded space by means of a scalar product $\langle v_1|v_2\rangle$, defining f as the adjoint of e. And we can see that $\langle v_j(h)|v_j(h)\rangle \approx \langle f^{2s+1-h}\cdot v_j(2s+1)|v_j(h)\rangle \approx \langle v_j(2s+1)|e\cdot v_j(2s)\rangle = 0, h \in (-2s, \ldots, 2s)$. In the decomposition at general q the vectors $v_j(h), h \in (-2s, \ldots, 2s)$ are orthogonal to the vectors, belonging to the representation V_{4s+1}. But now the pointed $v_j(h)$ are self-orthogonal and are linearly dependent on the vectors of V_{4s+1} with same values of h. As the orthogonal space of the non-zero vector $v_j(2s)$ contains itself already, there must exist a state $v'(s)$, with $h = 2s$, which is not orthogonal to $v_j(2s)$. It follows from $\langle v_j(2s+1)|e\cdot v'(2s)\rangle \approx \langle f\cdot v_j(2s+1)|v'(2s)\rangle \approx \langle v_j(2s)|v'(2s)\rangle \neq 0$ that $e\cdot v'(2s) = a v_j(2s+1)$ (a is a numerical non-zero coefficient). Solving the last equation, and then acting by f^{2s-h} on $v'(2s)$, we can find out the remaining states $v'(h), h \in (-2s, \ldots, 2s)$, which together with $v_j(h), h \in (-2j, \ldots, 2j)$ constitute the representation $\mathcal{I}^{(4(s+j)+2)}_{\{2j+1,2s+1\}}$ (see (56)).

So, under the conditions of statement II a modification of the decomposition rule (46) at roots of unity can take place characterized by the appearance of $\mathcal{I}$, which means that decomposition contains a representation with $j > j_{\max}$, and this in turn means fulfilment of the mentioned interrelated points. We see that the second and third points (which are equivalent to each other) provide necessary and sufficient criteria for such distortion, while the first one is only necessary.

The points of statement II for $V_{r_{\max}} \otimes V_3$ are considered in details in the appendix.

Statements I and II help us to determine all the possible $\mathcal{I}$-representations at $q^N = \pm 1$ and to formulate the modified fusion rules (see the following subsection).

Representation $\mathcal{I}$. Taking into account its origin from the fusion we can define *indecomposable representation* $\mathcal{I}^{(4(j_1+j_2)+2)}_{\{4j_2+1,4j_1+1\}}$ as a linear space $\{\upsilon(h), -2j_2 \leqslant h \leqslant 2j_2, \upsilon'(h'), -2j_1 \leqslant h' \leqslant 2j_1\}$, with the following action of the algebra generators:

$$\begin{cases} e\cdot\upsilon(h) = \beta_h^{j_2}(q)\upsilon(h+1), & e\cdot\upsilon(2j_1) = 0, e\cdot\upsilon(2j_2) = 0,\\ k\cdot\upsilon(h) = q^h\upsilon(h), & \\ f\cdot\upsilon(h) = \gamma_h^{j_2}(q)\upsilon(h-1), & f\cdot\upsilon(-2j_1) = 0, f\cdot\upsilon(-2j_2) = 0,\\ e\cdot\upsilon'(h') = \bar\beta_{h'}^{j_1}(q)\upsilon'(h'+1) + \tilde\beta_{h'}^{j_1}(q)\upsilon(h'+1), & \bar\beta_{2j_1}^{j_1}(q) = 0,\\ k\cdot\upsilon'(h) = q^{h'}\upsilon'(h'), & \\ f\cdot\upsilon'(h') = \bar\gamma_{h'}^{j_1}(q)\upsilon'(h'-1) + \tilde\gamma_{h'}^{j_1}(q)\upsilon(h'-1), & \bar\gamma_{-2j_1}^{j_1}(q) = 0, \end{cases} \tag{56}$$

with $\alpha^{j_2}_{2j_1+1}(q) = 0$, and at the same time $2j_1$ is the biggest h, for which $\alpha^{j_2}_{h+1}(q) = 0$. Hence the spins j_1 and j_2 are related by equations (52) and (53), which impose constraints on j_1 and j_2, in particular $2(j_2 - j_1) \geqslant 1$.

New functions $\bar\beta_h^{j_1}(q), \tilde\beta_h^{j_1}(q), \bar\gamma_h^{j_1}(q), \tilde\gamma_h^{j_1}(q)$ are constrained by the algebra relations, which give

$$\bar\beta_{h-1}^{j_1}\bar\gamma_h^{j_1} = \alpha_h^{j_1}, \qquad \bar\beta_h^{j_1}\tilde\gamma_{h+1}^{j_1} + \gamma_{h+1}^{j_2}\tilde\beta_h^{j_1} + \bar\gamma_h^{j_1}\tilde\beta_{h-1}^{j_1} + \beta_{h-1}^{j_2}\tilde\gamma_h^{j_1} = 0. \tag{57}$$

So, this representation has the structure described in (44). For general values of q the representation (56) would be, of course, completely reducible to the direct sum of the irreps V_{4j_1+1} and V_{4j_2+1}.

In figure 1(a), we presented a general representation $\mathcal{I}^{(4(j_1+j_2)+1)}_{\{4j_1+1,4j_2+1\}}$ diagrammatically, denoting by dots the states $\upsilon(h), \upsilon'(h)$ (the corresponding values of h are noted in the left column). The arrows ↑, ↖ show the action of the raising operator, while the arrows ↓, ↙ correspond to the action of the lowering operator. In the examples considered in section 3 the only transition (↖) we met was corresponding to the action $e \cdot \upsilon'(2j_1) = \tilde{\beta}^{j_1}_{2j_1}(q)\upsilon(2j_1+1)$. It is conditioned by the fact, that $\upsilon'(h)$-states, with $h = -2j_1+1, -2j_1+2, \ldots,$ were obtained by the action on the state $\upsilon'(-2j_1)$ of the operators e^p, $p = 1, \ldots, 4j_1$. Redefining states $\upsilon'(h)$ as $a\upsilon'(h) + b\upsilon(h)$ $(a, b \in \mathbb{Z})$, we should come to the more general case (56).

For a given $N, q^N = \pm 1$, the possible dimensions of the representations $\mathcal{I}^{(4(j_1+j_2)+2)}_{\{4j_2+1,4j_1+1\}}$ can be obtained from (52) and (53) with $j = j_2, h = 2j_1 + 1$: as the dimension of the representation (56) is $4(j_1 + j_2) + 2$, so for the integers N, p $(1 < Np \leqslant 2j_2)$, we obtain

$$\dim\left[\mathcal{I}^{(4(j_1+j_2)+2)}_{\{4j_2+1,4j_1+1\}}\right] = 4(j_1+j_2)+2 = \begin{cases} 4Np, & q^N = -1,\ N \text{ is even, and } q^N = 1,\ N \text{ is odd,} \\ 2Np, & q^N = -1,\ N \text{ is odd integer.} \end{cases} \tag{58}$$

For an illustration of the structure of $\mathcal{I}$ we can consider for example the indecomposable representation $\mathcal{I}^{(8)}_{\{5,3\}}$ at $q^4 = 1$ $(q^2 = -1)$ in a basis $\{\upsilon_h, \upsilon'_{h'}\}$ as follows:

$$e \cdot \{\upsilon_2, \upsilon_1, \upsilon_0, \upsilon_{-1}, \upsilon_{-2}, \upsilon'_1, \upsilon'_0, \upsilon'_{-1}\} = \{0, 0, -\mathrm{i}\upsilon_1, -\mathrm{i}\upsilon_0, -\mathrm{i}\upsilon_{-1}, -\upsilon_2, \upsilon'_1, \upsilon'_0\},$$
$$f \cdot \{\upsilon_2, \upsilon_1, \upsilon_0, \upsilon_{-1}, \upsilon_{-2}, \upsilon'_1, \upsilon'_0, \upsilon'_{-1}\} = \{-\mathrm{i}\upsilon_1, \mathrm{i}\upsilon_0, -\mathrm{i}\upsilon_{-1}, 0, 0, \upsilon'_0 + \upsilon_0, -\upsilon'_{-1} - \mathrm{i}\upsilon_{-1}, -\upsilon_{-2}\},$$
$$k \cdot \{\upsilon_2, \upsilon_1, \upsilon_0, \upsilon_{-1}, \upsilon_{-2}, \upsilon'_1, \upsilon'_0, \upsilon'_{-1}\} = \{-\upsilon_2, \mathrm{i}\upsilon_1, \upsilon_0, -\mathrm{i}\upsilon_{-1}, -\upsilon_{-2}, \mathrm{i}\upsilon'_1, \upsilon'_0, -\mathrm{i}\upsilon'_{-1}\}.$$

sdim$_q$. The notion of q-superdimension (for the non-graded algebras q-dimension) [3, 4, 12] of the representation V, $\mathrm{sdim}_q(V) = \mathtt{str}\, k$, where $\mathtt{str}$ denotes super-trace defined in the graded space of the representation, will be useful here. For the representations V_{4j+1} (or $\bar{V}_{4j+1}$)

$$\mathrm{sdim}_q(V_{4j+1}) = \sum_h (-1)^{p(\upsilon(h))} q^h = \begin{cases} \dfrac{q^{2j+1/2} + q^{-2j-1/2}}{q^{1/2} + q^{-1/2}}, & \text{if} \quad 4j+1 \quad \text{is odd,} \\ \dfrac{-q^{2j+1/2} + q^{-2j-1/2}}{q^{1/2} + q^{-1/2}}, & \text{if} \quad 4j+1 \quad \text{is even,} \end{cases} \tag{59}$$

where the sum goes over all the states labelled by h, and we assumed that the lowest weight vector has 0 parity. Let us also note that if $\mathrm{sdim}_q(V_{4j+1}) = 0$, then it follows that $(-q)^{4j+1} = 1$. So for the conventional odd r-dimensional representations, the relation $\mathrm{sdim}_q(V_r) = 0$ takes place, when $r = N, q^N = -1$ (and also $\mathrm{sdim}_q(\bar{V}_{pN}) = 0$), with odd integers N, p. And even-dimensional representations have 0 q-superdimension, $\mathrm{sdim}_q(V_r) = 0$ $(\mathrm{sdim}_q(\bar{V}_{pr}) = 0)$, if $r = 2N, q^N = \pm 1$, and N, p are integers.

It was stated that in the decompositions of tensor products an indecomposable representation appears instead of two representations *only* if the sum of their q-(super)dimensions is zero (see [3, 4]). The parities of the lowest weights of $V_{4j_i+1}, i = 1, 2$ in decompositions differ from one another by $[2(j_2 - j_1) \bmod 2]$. Taking this into account, one concludes that the relation

$$\left[\mathrm{sdim}_q(\bar{V}_{4j_2+1}) + \mathrm{sdim}_q(V_{4j_1+1})\right] = 0 \tag{60}$$

implies $[(q^{2(j_2+j_1)+1} + (-1)^{2(j_2+j_1)})(q^{2(j_1-j_2)} + (-1)^{2(j_1-j_2)}) = 0]$, which is in full agreement with the equalities (51) and (40), with $j = j_2, h = 2j_1 + 1$. So, relation (60) follows from statement II.

Note that the definition of the co-product of generator k implies that if one of the multipliers in the tensor product has vanishing q-superdimension, then the sum of the q-superdimensions over the representations in the decomposition is also equal to zero.

Remark. The representation, given by formulae (56), is indecomposable in general. However the structure $\mathcal{I} = \widehat{\bar{V} \oplus \bar{V}}$ having more than two lowest (highest) weights can be split into the sum of the indecomposable $\mathcal{I} = \widehat{\bar{V} \oplus V}$ ($p = 1$ in (58)) and the irreducible representations with $0q$-superdimension. It is conditioned by the appropriate values, which the coefficients β, γ can acquire in (56). Such a situation happens in the fusions $\otimes^n V_j$ of the irreps due to the nilpotency of the generators e, f (see sections 5.2 and 5.3). As we have seen in the discussed examples $\mathcal{I}^{(12)}_{\{7,5\}} = \widehat{\bar{V}_7 \oplus \bar{V}_5}$ splits into two $\mathcal{I}^{(6)}_{\{4,2\}}$-kind representations at $q^3 = -1$ (31), but it is not decomposable in (27). In the following discussion, we shall keep the notation $\mathcal{I}$ for the cases when $p > 1$ (58) too, recalling that in the fusions of the irreps they are decomposable.

5.2. Fusion rules

Here we intend to derive general fusion rules at roots of unity. As for the given value of q ($q^N = \pm 1$) the spin representations are no longer irreps starting from the spin value $\bar{j} = j_{\max} + \frac{1}{2}$ (with $j_{\max}$ being the maximal spin determined by (54) and (55)), then in the decomposition $V_{4j_1+1} \otimes V_{4j_2+1}$, at $(j_1 + j_2) \geqslant \bar{j}$, together with the allowed irreps, also indecomposable representations appear.

We can rewrite formula (58) to express the dimensions of $\mathcal{I}$-representations through the maximal dimension of the allowed irreps $r_{\max} = 4j_{\max} + 1$,

$$\dim[\mathcal{I}^{\mathcal{R}p}] = \begin{cases} 4Np = 4(2j_{\max}+1)p = (2r_{\max}+2)p, & q^N = -1,\ N \text{ even, and } q^N = 1,\ N \text{ odd}, \\ 2Np = 2(4j_{\max}+1)p = 2r_{\max}p, & q^N = -1,\ N \text{ is odd integer}. \end{cases} \tag{61}$$

Here $\mathcal{R} = 2N$ or $\mathcal{R} = 4N$ denotes the minimal dimension of $\mathcal{I}$-representations. Note that $\mathcal{N} = \mathcal{R}/2$.

V ⊗ V. It is evident from (61) that for $(-q)^{\mathcal{N}} = 1$ the representations $\mathcal{I}$, which appear in the tensor product of two arbitrary irreps $V_{r_1} \otimes V_{r_2}$ when $r_{\max} < (r_1 + r_2 - 1) \leqslant (2r_{\max} - 1)$, can be only with minimal dimensions ($\mathcal{I}^{\mathcal{R}p}$, $p = 1$), i.e. indecomposable. In the decomposition (16) for the general q the irrep with maximal dimension is $V_{r_1+r_2-1}$. For the exceptional values of q the representation $\bar{V}_{r_1+r_2-1}$ (and hence the remaining $\bar{V}_r$, $r_{\max} < r < r_1 + r_2 - 1$) cannot turn into the maximal sub-representation for $\mathcal{I}^{(2r_{\max}p)}$ or $\mathcal{I}^{2p(r_{\max}+1)}$, when $p > 1$.

When $(-q)^{\mathcal{N}} = 1$ for representation V_r from the interval $r_{\max} < r \leqslant r_1 + r_2 - 1$, it takes place $\alpha^{(r-1)/4}_{(\mathcal{R}-r+1)/2}(q) = 0$ (52), (53) and (61). Hence the points of statement II must be realized for the representations V_r and $V_{\mathcal{R}-r}$. From dimensional analysis it is clear that $r > \mathcal{R} - r$. In agreement with the conclusion of statement II all the representations $\bar{V}_r$, $r > r_{\max}$, starting from $\bar{V}_{r_1+r_2-1}$, are unifying with $V_{\mathcal{R}-r}$ to produce $\widehat{\bar{V}_r \oplus V_{\mathcal{R}-r}} = \mathcal{I}^{(\mathcal{R})}_{\{r,\mathcal{R}-r\}}$. The other V_{4j+1}-s, which do not coincide with $V_{\mathcal{R}-r}$, survive in this decomposition. By the solutions to the equation $e \cdot v'\left(\frac{\mathcal{R}-r-1}{2}\right) = v_{\frac{r-1}{4}}\left(\frac{\mathcal{R}-r+1}{2}\right)$, the states $\left\{v'\left(\frac{\mathcal{R}-r-1}{2}\right), f \cdot v'\left(\frac{\mathcal{R}-r-1}{2}\right), \ldots, f^{\mathcal{R}-r-1} \cdot v'\left(\frac{\mathcal{R}-r-1}{2}\right)\right\}$ can be constructed, which together with the states of representation $\bar{V}_r$ constitute the indecomposable representation $\mathcal{I}^{(\mathcal{R})}_{\{r,\mathcal{R}-r\}}$. In conclusion, we have

$$V_{r_1} \otimes V_{r_2} = \left(\bigoplus_{r=r_{\max}+1}^{r_1+r_2-1} \mathcal{I}^{(\mathcal{R})}_{\{r,\mathcal{R}-r\}} \right) \oplus \left(\bigoplus_{r' \leqslant r_{\max},\, r' \neq \{\mathcal{R}-(r_1+r_2-1),\ldots,\mathcal{R}-r_{\max}-1\}} V_{r'} \right). \tag{62}$$

This result is in the agreement with the rules derived by another technique (we are grateful to the author of [20] for the kind correspondence about this question). About the fusions of the irreps there was discussion in [9]. See also section 5.5 for a connection with the case of $sl_q(2)$.

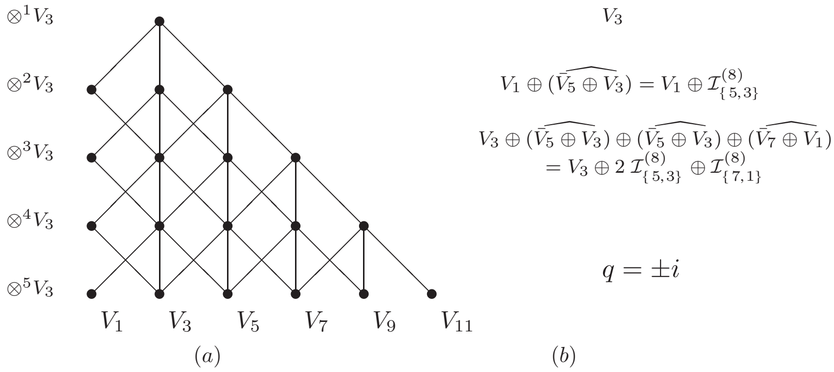

Figure 2. Bratteli diagram for the spin-$\frac{1}{2}$ irreps at general q (a) and fusion for $q^2 = -1$ (b).

$\otimes^{\mathbf{n}}\mathbf{V}$. The tensor product of the finite-dimensional representations of $\mathrm{osp}_q(1|2)$ for general q is reduced into a linear combination, and for n copies of the same representation we can write

$$\bigotimes^{n} V_{4j+1} = \underbrace{V_{4j+1} \otimes V_{4j+1} \otimes \cdots \otimes V_{4j+1}}_{n} = \bigoplus_{p=0}^{n/2} \varepsilon_n^{pj} V_{4p+1}, \tag{63}$$

where ε_n^{pj} stands for the multiplicity of the representation V_{4p+1} and can be calculated in easiest way using Bratteli diagrams (figure 2) [14]. In the left of nth row of the diagram denomination of tensor product $\otimes^n V_{4j+1}$ is placed, the representations V_{4p+1}, arising in (63) are denoted by dots which are located at the same row. The representations of the same type V_{4p+1}, regarding to different n-s, are arranged in vertical columns. The multiplicity ε_n^{pj} is determined by the number of all the possible paths leading from the top of the diagram to the given representation V_{4p+1} situated at the level n. The paths are formed by the lines connecting the dots. The intersections of two paths outside of the dots are to be ignored.

When q is a root of the unity, the decomposition (63) remains unchanged while $nj \leqslant 2j_{\max}$. We are interested in the representations emerging in the fusions of the fundamental irreps, i.e $j = 1/2$. The minimal n for which indecomposable representations appear in the decomposition of $\otimes^n V_3$, is $n = 2\bar{j} \equiv 2j_{\max} + 1$. The fusions corresponding to two possibilities (52) and (53) are respectively:

$$\otimes^{2\bar{j}} V_3 = \oplus_{j<\bar{j}} \left(\varepsilon_{2\bar{j}}^{j\ 3} - \delta_{j,\bar{j}-1/2}\right) V_{4j+1} + \mathcal{I}^{(4N)}_{\{2N+1,2N-1\}}, \tag{64}$$

$$\otimes^{2\bar{j}} V_3 = \oplus_{j<\bar{j}} \left(\varepsilon_{2\bar{j}}^{j\ 3} - \delta_{j,\bar{j}-1}\right) V_{4j+1} + \mathcal{I}^{(2N)}_{\{N+2,N-2\}}. \tag{65}$$

The associativity of the tensor product allows us to obtain this formula, using (62) for $V_{4j_{\max}+1} \otimes V_3$. From (54) and (55) it follows that $\bar{j} = N/2$ for the first case (54) and $\bar{j} = \frac{N+1}{4}$ for the second case (55). As in the decomposition of $\otimes^{2\bar{j}} V_3$ only the representation with maximal dimension $V_{4\bar{j}+1}$ becomes $\bar{V}_{4\bar{j}+1}$, then to reveal the structure of the possible indecomposable representations (with maximal proper sub-representation $\bar{V}_{4\bar{j}+1}$), which can appear in agreement with statement II, one has to check invariant sub-representations of $\bar{V}_{4\bar{j}+1}$. One can verify that the relations $\alpha^{\bar{j}}_{2\bar{j}}(q) = 0$ (54) and $\alpha^{\bar{j}}_{2\bar{j}-1}(q) = 0$ (55) take place, and the proper sub-representation of $\bar{V}_{4\bar{j}+1}$ is the representation spin-$(\bar{j} - 1/2)$ or spin-$(\bar{j} - 1)$ for the cases (54) or (55) correspondingly. In the fusions, the invariant sub-space of the representation $\bar{V}_{4\bar{j}+1}$ becomes linearly dependent on representation space of $V_{4\bar{j}-1}$ for the case (54) (correspondingly with $V_{4\bar{j}-3}$, for the case (55)), and then $\bar{V}_{4\bar{j}+1}$ together with other $4\bar{j}-1$ vectors (with $4\bar{j}-3$ vectors), forms a new $(4\bar{j}+1)+(4\bar{j}-1) = 4N$-dimensional indecomposable representation $\mathcal{I}^{(4N)}_{\{2N+1,2N-1\}}$ (64) $((4\bar{j}+1)+(4\bar{j}-3) = 2N$-dimensional indecomposable representation $\mathcal{I}^{(2N)}_{\{N+2,N-2\}}$ (65)). As the multiplicity of $V_{4\bar{j}+1}$ in the fusion is one, then in (64) and (65) the number of the indecomposable representations is also equal to 1. The multiplicities $\varepsilon^{j3}_{2\bar{j}}$ can be checked by means of Bratteli diagrams, as in the case of general q.

Now, let us present a scheme for derivation of fusion $\otimes^n V_3$ for an arbitrary n. To determine the decomposition of tensor product for the exceptional values of q, using (63) (defined for the general q), the following scheme can work: if $(nj) > j_{\max}$, the highest-dimensional representation V_{4nj+1} (appears in (63) with multiplicity $\varepsilon^{(nj)j}_n = 1$) turns to be $\bar{V}_{4nj+1}$. If $\mathrm{sdim}_q(\bar{V}_{4nj+1}) = 0$, this representation splits into the direct sum of the irreps V_r, with $\mathrm{sdim}_q(V_r) = 0$. It follows from the values of the enlarged center elements $e^{\mathcal{N}} = 0$, $f^{\mathcal{N}} = 0$ and the dimensional analysis before formula (60). Otherwise, if the largest invariant sub-representation of $\bar{V}_{4nj+1}$ is a $(4s+1)$-dimensional representation (i.e. s is the maximal $h/2$ for which $\alpha^{nj}_{h+1}(q) = 0$ in (48)), then the largest $\mathcal{I}$-representation arises from unification of $\bar{V}_{4nj+1}$ with one of V_{4s+1} appearing in the decomposition:

$$\widehat{\bar{V}_{4nj+1} \oplus \tilde{V}_{4s+1}} = \mathcal{I}^{(4(nj+s)+2)}_{\{4nj+1,4s+1\}}, \qquad \text{where} \quad \tilde{V}_{4s+1} = \begin{cases} \bar{V}_{4s+1} & \text{if} \quad s > j_{\max} \\ V_{4s+1} & \text{if} \quad s \leqslant j_{\max}. \end{cases}$$

Then one must consider in the same way the representation next to the highest dimensional, if it is not an irrep, i.e. $\bar{V}_{4nj-1}$, taking into account its multiplicity, which can be reduced by one, if $s = nj - \frac{1}{2}$, and so on.

As an example let us consider degeneracy of $\otimes^3 V_3 = V_1 \oplus 3V_3 \oplus 2V_5 \oplus V_7$ at $q = \pm \mathrm{i}$ (see figure 2). As $\alpha^{3/2}_1(\pm\mathrm{i}) = 0$, then $\bar{V}_7 \supset V_1$, so there is $\mathcal{I}^{(8)}_{\{7,1\}} = \widehat{\bar{V}_7 \oplus V_1}$. Then $\bar{V}_5 \supset V_3$, as $\alpha^1_2(\pm i) = 0$, and two $\bar{V}_5$ become the part of two $\mathcal{I}^{(8)}_{\{5,3\}} = \widehat{\bar{V}_5 \oplus V_3}$. So we have $V_3 \oplus 2\mathcal{I}^{(8)}_{\{5,3\}} \oplus \mathcal{I}^{(8)}_{\{7,1\}}$. For arbitrary n at $q = \pm\mathrm{i}$ moving in the same way, relation (35) can be traced, finding multiplicities from the dimensional analysis. For $\otimes^4 V_3$, at $q^3 = -1$, the analysis gives $\mathrm{sdim}_q(V_3) = 0$, $\mathrm{sdim}_q(\bar{V}_9) = 0$, as $e^3 = 0$, $f^3 = 0$. Hence $\bar{V}_9 = \oplus^3 V_3$ and $\otimes^4 V_3 = 9V_3 \oplus 3\big(\mathcal{I}^{(12)}_{7,5}\big) \oplus 3\big(\mathcal{I}^{(6)}_{5,1}\big) = 9V_3 \oplus 6\big(\mathcal{I}^{(6)}_{4,2}\big) \oplus 3\big(\mathcal{I}^{(6)}_{5,1}\big)$.

As was noted, the representation $\mathcal{I}^{(p\mathcal{R})} = \widehat{\bar{V}_r \oplus \bar{V}_{p\mathcal{R}-r}}$, $p > 1$, arising in the fusions $\otimes^n V_3$, is decomposable. The $\mathcal{N}$-nilpotency of e, f $(\mathcal{N} = \mathcal{R}/2)$ helps us to determine how many and what kind of irreducible invariant sub-representations has the proper sub-representation $\bar{V}_r$ of $\mathcal{I}^{(p\mathcal{R})}_{\{r,p\mathcal{R}-r\}}$. One can directly count that $\bar{V}_r$ contains p proper irreducible subspaces with dimension $\mathbf{r_p}$,

$$p = [r/\mathcal{N}], \qquad \mathbf{r_p} = \mathcal{N} - r + \mathcal{N}p, \tag{66}$$

(here $[x]$ denotes the integer part of x). So, $\bar{V}_r \supset \underbrace{V_{\mathbf{r_p}} \oplus V_{\mathbf{r_p}} \cdots \oplus V_{\mathbf{r_p}}}_{p}$, $\mathbf{r_p} \leqslant 4j_{\max} + 1$. The highest weight vectors of $\bar{V}_r$, together with $v_j(j)$, $j = (r-1)/4$), now are $v_j(2j^i)$, $2j^i = -2j + i\mathcal{R}/2 - 1 \equiv (i\mathcal{R} - r - 1)/2$, $i = 1, \ldots, p$. The linear dependence can be established between $p\mathbf{r_p}$ vectors belonging to the mentioned proper irreducible subspaces of $\bar{V}_r$ and the corresponding vectors with the same weights of the representation $\bar{V}_{p\mathcal{R}-r}$. All these vectors have null norms. And new vectors $v'(h)$ not orthogonal to them are to be constructed similar to the case when $p = 1$. So, the internal structure of $\mathcal{I}^{(p\mathcal{R})}_{\{r,p\mathcal{R}-r\}}$ is characterized as

$$\mathcal{I}^{(p\mathcal{R})}_{r,p\mathcal{R}-r} = \underbrace{\mathcal{I}^{(\mathcal{R})}_{\{\mathcal{R}-\mathbf{r_p},\mathbf{r_p}\}} \oplus \mathcal{I}^{(\mathcal{R})}_{\{\mathcal{R}-\mathbf{r_p},\mathbf{r_p}\}} \cdots \oplus \mathcal{I}^{(\mathcal{R})}_{\{\mathcal{R}-\mathbf{r_p},\mathbf{r_p}\}}}_{p}. \tag{67}$$

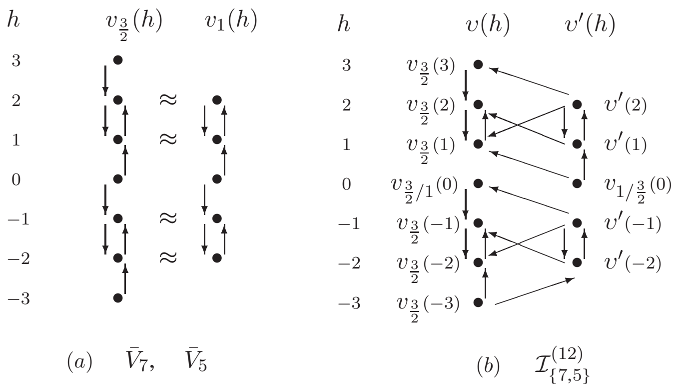

Figure 3. Representations $\bar{V}_7$, $\bar{V}_5$ (*a*) and $\mathcal{I}^{(12)}_{\{7,5\}}$ (*b*) in $\otimes^3 V_3$, $q^3 = -1$.

The sign '=' in (67) means an isomorphism, as the eigenvalues of the generator k on the states in rhs differ by common multipliers from those of $\mathcal{I}^{(\mathcal{R})}_{\{\mathcal{R}-\mathbf{r_p},\mathbf{r_p}\}} = \widehat{\bar{V}_{\mathcal{R}-\mathbf{r_p}} \oplus V_{\mathbf{r_p}}}$.

For illustration we represent the emergence of $\mathcal{I}^{(12)}_{\{7,5\}}$ at $q^3 = -1$ in the decomposition of $V_3 \otimes V_3 \otimes V_3$. In figure 3(*a*), two non-completely reducible representations $\bar{V}_7$, $\bar{V}_5$ are shown, which have two proper two-dimensional subspaces with $h = 1, 2$ and $h = -1, -2$. There is a linear dependence between the corresponding vectors $v_{\frac{3}{2}}(h)$ and $v_1(h)$, which in the figure is denoted by the symbol '≈'. In figure 3(*b*), the structure of $\mathcal{I}^{(12)}_{\{7,5\}} = \widehat{\bar{V}_7 \oplus \bar{V}_5}$ is presented. This representation is decomposed into two indecomposable representations $\mathcal{I}^{(6)} = \widehat{\bar{V}_4 \oplus V_2}$. By means of $v_{\frac{3}{2}/1}(0)$ and $v_{1/\frac{3}{2}}(0)$ two mutually orthogonal vectors are denoted, which are some linear superpositions of the vectors $v_{\frac{3}{2}}(0)$ and $v_1(0)$.

As an another example let us observe the case $q^2 = -1$, which will give the exact decomposition of (35). $\mathcal{I}^{(p8)}$ can be composed by $\widehat{\bar{V}_{4p+1} \oplus \bar{V}_{4p-1}}$ or $\widehat{\bar{V}_{4p+3} \oplus \bar{V}_{4p-3}}$ (consideration of the even-dimensional irreps would enlarge the possibilities by $\widehat{\bar{V}_{4p+2} \oplus \bar{V}_{4p-2}}$). It can be checked straightforwardly (66) that $\bar{V}_{4p+1}$ ($\bar{V}_{4p+3}$) has V_3 (V_1)-type invariant sub-irreps. It gives

$$\mathcal{I}^{(8p)}_{\{4p+1,4p-1\}} = \underbrace{\mathcal{I}^{(8)}_{\{5,3\}} \oplus \mathcal{I}^{(8)}_{\{5,3\}} \oplus \cdots \mathcal{I}^{(8)}_{\{5,3\}}}_{p}, \qquad \mathcal{I}^{(8p)}_{\{4p+3,4p-3\}} = \underbrace{\mathcal{I}^{(8)}_{\{7,1\}} \oplus \mathcal{I}^{(8)}_{\{7,1\}} \oplus \cdots \mathcal{I}^{(8)}_{\{7,1\}}}_{p}. \tag{68}$$

$\mathbf{V} \otimes \mathcal{I}, \mathcal{I} \otimes \mathcal{I}$. The fusion rules of the products like $V \otimes \mathcal{I}$ and $\mathcal{I} \otimes \mathcal{I}$ can be found either from the decomposition of $\otimes^n V$, recalling the associativity property of the product (quite analogous to the cases (32), (33) and (34)), or in this way: let $\mathcal{I} = \widehat{V' \oplus V''}$, then one must write down the tensor product for $V' \oplus V''$ at general q, and analyze its deformation at the exceptional values quite similar to the case $\otimes^n V$. Let us present the decomposition of $\mathcal{I}^{(\mathcal{R})}_{\{r,\mathcal{R}-r\}} \otimes V_3 = (\widehat{\bar{V}_r \oplus V_{\mathcal{R}-r}}) \otimes V_3$. From the fusion rule at general q (we assume that $\mathcal{R} - r > 1$)

$$(V_r \oplus V_{\mathcal{R}-r}) \otimes V_3 = V_{r+2} \oplus V_r \oplus V_{r-2} \oplus V_{\mathcal{R}-r+2} \oplus V_{\mathcal{R}-r} \oplus V_{\mathcal{R}-r-2}, \tag{69}$$

such decomposition at roots of unity ($((-q)^{\mathcal{R}/2}=1$, see (61)) will be followed:

$$\mathcal{I}^{(\mathcal{R})}_{\{r,\mathcal{R}-r\}}\otimes V_3=\widehat{\bar{V}_{r+2}\oplus V_{\mathcal{R}-r-2}}\oplus\widehat{\bar{V}_r\oplus V_{\mathcal{R}-r}}\oplus\widehat{\tilde{V}_{r-2}\oplus\tilde{V}_{\mathcal{R}-r+2}}$$
$$=\mathcal{I}^{(\mathcal{R})}_{\{r+2,\mathcal{R}-r-2\}}\oplus\mathcal{I}^{(\mathcal{R})}_{\{r,\mathcal{R}-r\}}\oplus\begin{cases}V_{r_{\max}}\oplus V_{r_{\max}}, & \text{if}\quad r-2=r_{\max},\quad \mathcal{R}/2\ \text{odd}\\ \mathcal{I}^{(\mathcal{R})}_{\{r_{\max}+2,r_{\max}\}}, & \text{if}\quad r-2=r_{\max},\quad \mathcal{R}/2\ \text{even}\\ \mathcal{I}^{(\mathcal{R})}_{\{r-2,\mathcal{R}-r+2\}}, & \text{if}\quad r-2>r_{\max}.\end{cases}\tag{70}$$

5.3. Clebsh–Gordan coefficients

To make sure by *direct construction* that in the decomposition of the tensor products two representations at $q^N=\pm1$ are unified in the manner described above (statement II), $\widehat{\bar{V}_{4j+1}\oplus V_{4s+1}}$, one can calculate Clebsh–Gordan coefficients for the representations of this algebra and check the linear dependence of the vectors belonging to $\bar{V}_{4j+1}$ and V_{4s+1}.

Let us recall the definition of Clebsh–Gordan (CG) coefficients: if $V_{4j+1}=\{v_j(h)\}$, $h=-2j,\ldots,2j$, is an irrep arising in the decomposition (46), then its states are defined as

$$v_j(h)=\sum_{h_1+h_2=h}C\begin{pmatrix}j_1, j_2, j\\ h_1, h_2, h\end{pmatrix}v_{j_1}(h_1)\otimes v_{j_2}(h_2).\tag{71}$$

The second point of statement II affirms that in the fusions two representations $V_{4j+1}, V_{4s+1}, j>s$ are replaced by an indecomposable one, when V_{4s+1} and a sub-representation of $V_{4j+1}, \{v_j(h)\}$ with $h=-2s,\ldots,2s$, are linearly dependent, i.e $\{v_j(-2s),\ldots,v_j(2s)\}\approx\{v_s(-2s),\ldots,v_s(2s)\}$. In terms of Clebsh–Gordan coefficients it means that

$$\left\{C\begin{pmatrix}j_1, j_2, j\\ h_1, h_2, h\end{pmatrix}\right\}_{h_1+h_2=h}\approx\left\{C\begin{pmatrix}j_1, j_2, s\\ h_1, h_2, h\end{pmatrix}\right\}_{h_1+h_2=h},\qquad h=-2s,\ldots,2s.\tag{72}$$

This relation implies that the functions $C\binom{j_1,j_2,j}{h_1,h_2,h}/C\binom{j_1,j_2,s}{h_1,h_2,h}$ do not depend on the variables h_1, h_2.

Here we calculate the coefficients up to the normalization factors, which are inessential when q is given by a root of unity. Using the highest weight method [23] and the co-product (8) for the representations (47) we find the following expressions for $C\binom{j_1,j_2,j}{h_1,h_2,h}$, $h=2j$,

$$C\begin{pmatrix}j_1, j_2, j\\ h_1, 2j-h_1, 2j\end{pmatrix}=\prod_{g=h_1+1}^{2j_1}\left(\frac{(-1)^{p_{j_1,g+1}}q^{2j-g+1}\beta^{j_2}_{2j-g}(q)}{\beta^{j_1}_{g-1}(q)}\right)C\begin{pmatrix}j_1, j_2, j\\ 2j_1, 2j-2j_1, 2j\end{pmatrix}.\tag{73}$$

The parity p_{j_1,h_1} of the state $v_{j_1}(h_1)$ can be determined in this way: if the lowest weight vectors in the rhs of (71) have even parity, then $(-1)^{p_{j_1,h_1}}=(-1)^{2j_1+h_1}$. Acting by the operator f^{2j-h} on the both sides of equation (71), where $v_j(2j)$ stands on the lhs, we arrive at

$$C\begin{pmatrix}j_1, j_2, j\\ h_1, h_2, h\end{pmatrix}=\prod_{g=h+1}^{2j}\left(\gamma^j_g(q)\right)^{-1}\sum_{r=0}^{2j-h}\begin{bmatrix}2j-h\\ r\end{bmatrix}_{-q^{-1}}(-1)^{rp_{j_1,h'_1}}q^{r(h'_1)}$$
$$\times\prod_{r_1=h'_1}^{h_1+1}\gamma^{j_1}_{r_1}(q)\prod_{r_2=h'_2}^{h_2+1}\gamma^{j_2}_{r_2}(q)C\begin{pmatrix}j_1, j_2, j\\ h'_1, h'_2, 2j\end{pmatrix},\tag{74}$$

where $h'_1=h_1+(2j-h-r)$, $h'_2=2j-h'_1=h_2+r$, and p_{j_1,h'_1} is the parity of the state $v_{j_1}(h'_1)$. The q-binomial coefficients are defined by formula (11). We fix the coefficients up to a normalization constant, as for q being a root of unity, their ratios become important rather than coefficients themselves.

Note that in different works there are computed formulae for Clebsh–Gordan coefficients with fixed quantities β, γ [4, 19]; particularly in the work [4] a specific case $j_2 = 1/2, C\binom{j_1, 1/2, j}{h_1, h_2, h}$ is given, which coincides with our computations up to the normalization factors.

If one checks relations (72) directly using formulae (74), then one has to remove all the possible common zeroes and singularities appearing in the coefficients of the vectors $v_j(h)$ $(v_s(h))$ (71) at the corresponding exceptional values of q and to verify that the ratios $C\binom{j_1, j_2, j}{h_1, h_2, h} / C\binom{j_1, j_2, j}{\tilde{h}_1, \tilde{h}_2, h}$ coincide with $C\binom{j_1, j_2, s}{h_1, h_2, h} / C\binom{j_1, j_2, s}{\tilde{h}_1, \tilde{h}_2, h}$, $h_1 + h_2 = \tilde{h}_1 + \tilde{h}_2 \equiv h \in -2s, \ldots, 2s$. The quantities β_h^j, γ_h^j can be defined as $\beta_h^j = 1, \gamma_h^j = \alpha_h^j$ for the allowed irreps. One must take into account that $\alpha_h^j(q) = \alpha_h^s(q)$, which follows from $(-q)^{2j+2s+1} = 1$ (48). See the appendix for the case of $V_{j_{\max}} \otimes V_3$.

On the other hand when $j > j_{\max}$ and $\alpha_{2s+1}^j(q) = 0$, i.e. $(-q)^{2s+2j+1} = 1$, then the vector $v_j(2s)$ is also a highest weight vector . Hence the highest weight method can be applied also for this vector to find the ratios of its CG coefficients. We can write formulae similar to (73) and (74), replacing $2j$ by $2s$ (below we suppose $s \geqslant j_1$)

$$C\begin{pmatrix} j_1, j_2, j \\ h_1, 2j - h_1, 2s \end{pmatrix} = \prod_{g=h_1+1}^{2j_1} \left(\frac{(-1)^{p_{j_1,g+1}} q^{2s-g+1} \beta_{2s-g}^{j_2}}{\beta_{g-1}^{j_1}} \right) C\begin{pmatrix} j_1, j_2, j \\ 2j_1, 2s - 2j_1, 2s \end{pmatrix}, \tag{75}$$

$$C\begin{pmatrix} j_1, j_2, j \\ h_1, h_2, h \end{pmatrix} = \prod_{g=h+1}^{2s} \left(\gamma_g^j(q)\right)^{-1} \sum_{r=0}^{2s-h} \begin{bmatrix} 2s - h \\ r \end{bmatrix}_{-q^{-1}} (-1)^{r p_{j_1, h_1'}} q^{r(h_1')}$$
$$\times \prod_{r_1 = h_1'}^{h_1+1} \gamma_{r_1}^{j_1}(q) \prod_{r_2 = h_2'}^{h_2+1} \gamma_{r_2}^{j_2}(q) C\begin{pmatrix} j_1, j_2, j \\ h_1', h_2', 2s \end{pmatrix}, \tag{76}$$

where now $h' - 1 = h_1 + (2s - h - r), h_2' = 2s - h_1', -2s \leqslant h < 2s$. Comparing these expressions with $C\binom{j_1, j_2, s}{h_1, 2j - h_1, h}$, obtained from formulae (73) and (74) , we see that it ensures the validity of relations (72): $\frac{C\binom{j_1, j_2, j}{h_1, 2j-h_1, 2s}}{C\binom{j_1, j_2, s}{h_1, 2j-h_1, 2s}} = \frac{C\binom{j_1, j_2, j}{2j_1, 2s-2j_1, 2s}}{C\binom{j_1, j_2, s}{2j_1, 2s-2j_1, 2s}}$, $\frac{C\binom{j_1, j_2, j}{h_1, 2j-h_1, h}}{C\binom{j_1, j_2, s}{h_1, 2j-h_1, h}} = \prod_{g=h+1}^{2s} \frac{\gamma_g^s(q)}{\gamma_g^j(q)} \frac{C\binom{j_1, j_2, j}{2j_1, 2s-2j_1, 2s}}{C\binom{j_1, j_2, s}{2j_1, 2s-2j_1, 2s}}$, $-2s \leqslant h < 2s$.

In the decomposition (62) for the irreps $V_{r'}$ the coefficients of the expansion (71) are to be obtained just from formulae (74), fixing the values of q. For the indecomposable representations $\mathcal{I}_{\{r, \mathcal{R}-r\}}^{(\mathcal{R})} = \{v_{\mathcal{I}}(-2j), \ldots, v_{\mathcal{I}}(2j); v_{\mathcal{I}}'(-2s), \ldots, v_{\mathcal{I}}'(2s)\}, r = 4j+1, \mathcal{R}-r = 4s + 1, (-q)^{2j+2s+1} = 1$

$$v_{\mathcal{I}}(h) = \sum_{h_1+h_2=h} C_{\mathcal{I}}\begin{pmatrix} j_1, j_2, j \\ h_1, h_2, h \end{pmatrix} v_{j_1}(h_1) \otimes v_{j_2}(h_2), \tag{77}$$

$$v_{\mathcal{I}}'(h) = \sum_{h_1+h_2=h} C_{\mathcal{I}}'\begin{pmatrix} j_1, j_2, s \\ h_1, h_2, h \end{pmatrix} v_{j_1}(h_1) \otimes v_{j_2}(h_2), \tag{77}$$

the coefficients $C_{\mathcal{I}}\binom{j_1, j_2, j}{h_1, h_2, h}$ for the vectors $v_{\mathcal{I}}(h)$ also can be calculated from (74) in the limit when q is a root of unity (only one must be careful, as now for the values $h < -2s$ there are common overall factors like $[2j + 2s + 1]_{(-q)^{1/2}}$ (11) which are cancelled by the choice

$\gamma^j_{2s+1}(q) = \alpha^j_{2s+1}(q)$). As for the vectors $v'_{\mathcal{I}}(h)$, here the coefficients can be obtained using the relation $e \cdot v'_{\mathcal{I}}(2s) = \tilde{\beta}^s_{2s}(q) v_{\mathcal{I}}(2s+1)$ (56). The resulting expression for the coefficients of $v'_{\mathcal{I}}(2s)$ is

$$C'_{\mathcal{I}}\begin{pmatrix} j_1, j_2, s \\ h_1, h_2, 2s \end{pmatrix} = \prod_{i=h_1}^{2j_1-1} f_s(j_1, j_2, i) C'_{\mathcal{I}}\begin{pmatrix} j_1, j_2, s \\ 2j_1, 2s-2j_1, 2s \end{pmatrix} + \frac{\tilde{\beta}^s_{2s}(q)}{f_s(j_1, j_2, h_1)} \sum_{i'=h_1}^{2j_1-1} \prod_{i=h_1}^{i'} f_s(j_1, j_2, i) \frac{q^{2s-i'}}{\beta^{j_1}_{i'}(q)} C_{\mathcal{I}}\begin{pmatrix} j_1, j_2, j \\ i'+1, 2s-i', 2s+1 \end{pmatrix}, \quad (78)$$

where $f_s(j_1, j_2, i) = (-1)^{p_{j_1,i+2}} q^{2s-i} \frac{\beta^{j_2}_{2s-i-1}(q)}{\beta^{j_1}_i(q)}$ is a function entering into formulae (73) and (75). Hence the first summand corresponds to the solution of the homogeneous equation. The indefiniteness of that term (coming from $C'_{\mathcal{I}}\binom{j_1, j_2, s}{2j_1, 2s-2j_1, 2s}$) can be absorbed by redefinition of the vectors $v'(h)$, which are determined up to a solution to the homogeneous equation $e \cdot v(h) = 0$. With the operator f^{2s-h} acting on $v'(2s)$ it is possible to find out the coefficients corresponding to the remaining vectors $v'_{\mathcal{I}}(h)$. We obtain the following expression for $C'_{\mathcal{I}}$ (below $h'_1 = h_1 + (2s - h - r)$, $h'_2 = 2s - h'_1$):

$$C'_{\mathcal{I}}\begin{pmatrix} j_1, j_2, s \\ h_1, h_2, h \end{pmatrix} = \prod_{g=h+1}^{2s} \left(\bar{\gamma}^j_g(q)\right)^{-1} \sum_{r=0}^{2s-h} \begin{bmatrix} 2s-h \\ r \end{bmatrix}_{-q^{-1}} (-1)^{r p_{j_1,h'_1}} q^{r(h'_1)} \times \prod_{r_1=h'_1}^{h_1+1} \gamma^{j_1}_{r_1}(q) \prod_{r_2=h'_2}^{h_2+1} \gamma^{j_2}_{r_2}(q) C'_{\mathcal{I}}\begin{pmatrix} j_1, j_2, s \\ h'_1, h'_2, 2s \end{pmatrix}, \quad (79)$$

up to the additive term like $C_{\mathcal{I}}\binom{j_1, j_2, j}{h_1, h_2, h} P\left(\gamma^s_i, \bar{\gamma}^s_{i'}, \tilde{\gamma}^s_{i''}\right)$, where $P\left(\gamma^s_i, \bar{\gamma}^s_{i'}, \tilde{\gamma}^s_{i''}\right)$ is a rational function.

6. Even-dimensional (unconventional) representations and the quantum algebras osp$_q$(1|2) and $s\ell_t(2)$ at roots of unity

To complete our analysis we observe also the emerging of the indecomposable representations in the fusions for the even-dimensional irreducible representations, which have no classical counterparts (see section 2). Let us denote $V_r = \{(1-r)/2 + (\imath\pi/(2\log q)), (3-r)/2 + (\imath\pi/(2\log q)), \ldots, (r-1)/2 + (\imath\pi/(2\log q))\}$ by V_{4j+1}, with $2j = \frac{r-1}{2}$.

In accordance with (16), for the general values of q, decomposition of the tensor products for the representations with odd and even dimensions has form (46), but now with j it takes integer, half-integer or quarter integer values.

Representation $V_r, r \in 2\mathbb{Z}_+$. Let us present the action of the algebra on the vectors $\{v_j(h)\}, h = -2j+\lambda, -2j+1+\lambda, \ldots, 2j+\lambda$, of the *even-dimensional irreducible representation* V_{4j+1} (j is a quarter integer) as follows:

$$\begin{cases} k \cdot v_j(h) = q^h v_j(h), & \\ e \cdot v_j(h) = v_j(h+1), & e \cdot v_j(2j+\lambda) = 0, \\ f \cdot v_j(h) = \alpha^j_h(q) v_j(h-1), & f \cdot v_j(-2j+\lambda) = 0, \end{cases} \quad (80)$$

$$\alpha^j_h(q) = \sum_{i=0}^{2j-h+\lambda} (-1)^{2j-(h-\lambda)-i} [2j+\lambda-i]_q, \qquad h = -2j+1+\lambda, \ldots, 2j+\lambda. \quad (81)$$

Here in comparison with the previous cases (47) and (56), we have specified parameters β and γ.

Recall that irreducibility of V_{4j+1} turns out to be spoiled, if at least one of the functions $\alpha_h^j(q)$ vanishes, indicating the existence of a proper sub-representation.

In the case under consideration (81), taking into account that $q^{2\lambda} = -1$, the equation $\alpha_h^j(q) = 0$ is equivalent to (below the notation $\bar{h} = h - \lambda$ is used)

$$(1 - (-1)^{2j+\bar{h}} q^{2j+\bar{h}})(1 + (-1)^{2j-\bar{h}} q^{2j+1-\bar{h}}) = 0. \tag{82}$$

It follows from the property $\alpha_{r+\lambda}^j(q) = -\alpha_{-r+\lambda+1}^j(q)$ that two multipliers on the lhs of (82) are equivalent. Hence one can consider only one of them, say $(1 - (-1)^{2j+\bar{h}} q^{2j+\bar{h}}) = 0$, which has solution of the form:

$$\text{If} \quad q^{2n} = 1, \quad n \in \mathbb{N}, \quad \text{then} \quad \bar{h} = 2np - 2j, \quad p \in \mathbb{N}, \quad 0 < 2np \leqslant 4j. \tag{83}$$

$$\text{If} \quad q^{2n+1} = -1, \quad n \in \mathbb{N}, \quad \text{then} \quad \bar{h} = (2n+1)p - 2j, \quad p \in \mathbb{N} \quad 0 < (2n+1)p \leqslant 4j. \tag{84}$$

If j_1 is the weight $\bar{h}/2$ corresponding to $\alpha_{h+1}^j(q) = 0$, for which $|h - \lambda|$ takes its maximal value, then $\bar{V}_{4j+1}$ has proper sub-representation V_{4j_1+1}. According to the above observations (see statement II), in the fusions such representations $\widehat{\bar{V}_{4j+1} \oplus V_{4j_1+1}} = \mathcal{I}_{\{4j+1,4j_1+1\}}^{4(j+j_1)+2}$ can appear.

It follows from (83) and (84) that the dimensions of $\mathcal{I}$-representations are again defined by formula (58). Their structures are described in (56). An example is the indecomposable representation at $q^3 = -1$, $\mathcal{I}_{\{4,2\}}^{(6)} = \widehat{\bar{V}_4 \oplus V_2}$, studied in detail in the third subsection (20). So for the given $n, q^n = \pm 1$, inclusion of the even-dimensional irreps could enlarge the class of the representations $\mathcal{I}^{2np}$ with representations $\mathcal{I}_{\{2r,2np-2r\}}^{(2np)}$, which can appear in the mixed fusions as $V_{2s_1} \otimes V_{2s_2+1}$.

Note. There is an interesting fact, which we would like to mention. As was stated, the odd-dimensional irreps of $\mathrm{osp}_q(1|2)$ form a closed fusion at general q. However for the cases when q is a root of unity and $\lambda = \imath\pi/(2\log q)$ is a rational quantity, in the decomposition of the multiple tensor products of the conventional irreps such indecomposable representations can arise, which have even-dimensional irreps' origin: in (66) $\mathbf{r_p}$ accepts an even number, when $\mathcal{N}$ is odd and p is even. In the mentioned cases in $\otimes^n V_3$ a direct sum $V_r \oplus V_{p\mathcal{R}-r}$ at $(-q)^{\mathcal{R}/2} = 1$ $(r > r_{\max})$ replaces with $p\big(\widehat{\bar{V}_{\mathcal{R}-\mathbf{r}_p} \oplus V_{\mathbf{r}_p}}\big)$, and while r, $p\mathcal{R} - r$ are odd numbers, the dimensions $\mathbf{r}_p$ and $\mathcal{R} - \mathbf{r}_p$ are even.

The resemblance of the representations of the algebras $\mathrm{osp}_q(1|2)$ *and* $s\ell_t(2)$ *at roots of unity.* The correspondence between the odd-dimensional conventional irreps of the quantum superalgebra $\mathrm{osp}_q(1|2)$ and the odd-dimensional non-spinorial irreps of the quantum algebra $s\ell_t(2)$ $(t = \mathrm{i}q^{1/2})$ was mentioned and investigated in the works [4, 18]. The consideration of the even-dimensional irreps of $\mathrm{osp}_q(1|2)$ [15] made the equivalence of the finite-dimensional irreducible representations of $\mathrm{osp}_q(1|2)$ and $s\ell_t(2)$ at general q complete. About the correspondence of the R-matrices and Lax operators with symmetry of $\mathrm{osp}_q(1|2)$ and $s\ell_t(2)$ there are discussions in [4, 15, 17].

The classification and investigation of the finite-dimensional representations of $s\ell_t(2)$, when t is a root of unity can be found in the works [3, 7]. For any $\mathcal{N}_t$ $\left[\mathcal{N}_t = \left\{\begin{smallmatrix} N/2, \text{ even } N \\ N, \text{ odd } N \end{smallmatrix}\right\}\right.$,

$t^N = 1\big]$, the lowest weight indecomposable representation, emerging from the fusions of the irreps, has dimension $2\mathcal{N}_t$.

The relation $t^{2N} = (-1)^N q^N$ helps us to connect the dimensions of the indecomposable representation $\mathcal{I}^{\mathcal{R}}$ of the algebra $\mathrm{osp}_q(1|2)$ with the respective ones of $s\ell_t(2)$. The first possibility in (61) corresponds to the relation $t^{2N} = -1$, i.e. $\mathcal{N}_t = 2N$, the second one corresponds to $t^{2N} = 1$, i.e. $\mathcal{N}_t = N$. This means that the dimension of the representation $\mathcal{I}^{\mathcal{R}}$ can also be presented as $2\mathcal{N}_t$. In the same way the correspondence of the dimensions of the permissible irreps can be stated.

As we see, the mentioned equivalence of the representations of two quantum algebras can be extended also for the exceptional values of q [4]. All the tools and principles which are used in this paper (Clebsh–Gordan decomposition, statements and so on) are valid also in the case of the algebra $s\ell_t(2)$. This similarity can help us to compare the analysis of the fusion rules when q is a root of unity with the known schematic results [7, 12] and to be convinced of their correspondence, and also to extend the detailed analysis of the fusion rules of the multiple tensor products of the irreps and indecomposable representations to the case of $s\ell_t(2)$.

7. Summary

We studied the lowest weight representations of $\mathrm{osp}_q(1|2)$ at *the exceptional values of q* (when q is a root of unity), and as a result we listed all the possible irreps and indecomposable representations appearing for given $N, q^N = \pm 1$, and formulated the modification of the conventional fusion rules. We described how and when indecomposable representations appear in the decompositions of the tensor products. It led to a scheme for explicit construction of the decompositions for the tensor products of both irreducible and indecomposable representations when the deformation parameter takes exceptional values.

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Appendix. The tensor product decompositions by direct construction

Equations (64) and (65). Here we would like to return once again to formulae (64) and (65). Due to associativity of the co-product, the representation of type $\mathcal{I}$ can arise only from the degeneration of the decomposition of the following tensor product (below we use notation $j_{\max} \equiv J$):

$$V_{4J+1} \otimes V_3 = V_{4J-1} \oplus V_{4J+1} \oplus V_{4J+3}. \tag{A.1}$$

Let us denote the vector states of the irreps on the lhs of (A.1) by $u_J(h)$ and $u_{\frac{1}{2}}(h)$. After calculation of the corresponding coefficients (74) and inserting them in (71), one obtains the following expressions (up to the common multipliers) for the vector states $v_j(2J)$ of the representations on the rhs of equation (A.1)

$$v_{J+\frac{1}{2}}(2J) = [2J]_q\big(u_J(2J-1) \otimes u_{\frac{1}{2}}(1)\big) + q^{2J}\big(u_J(2J) \otimes u_{\frac{1}{2}}(0)\big), \tag{A.2}$$

$$v_J(2J) = \big(u_J(2J-1) \otimes u_{\frac{1}{2}}(1)\big) - q^{-1}\big(u_J(2J) \otimes u_{\frac{1}{2}}(0)\big). \tag{A.3}$$

So, recalling (54), one sees that for the cases $q^N = 1$, with odd integer N, or $q^N = -1$, with even integer N, the relation $v_{J+\frac{1}{2}}(2J) = (-1)^N v_J(2J)$ takes place. The relations among the remaining vectors $v_j(h)$ can be obtained by the repeated actions of the lowering operator f. As the action of the operators $f^n, n < 4J$, on the states $v_{J+\frac{1}{2}}(2J)$ and $v_J(2J)$ does not annihilate them (V_{4J+1} is an irreducible representation), we can take $\gamma_h^{J+\frac{1}{2}}(q) = 1, \gamma_h^J(q) = 1, h = \{-2J+1, \ldots, 2J\}$. Then

$$\left\{v_{J+\frac{1}{2}}(2J), \ldots, v_{J+\frac{1}{2}}(-2J)\right\} = (-1)^N\{v_J(2J), \ldots, v_J(-2J)\}. \tag{A.4}$$

In the same way we check, that the vectors $v_{J+\frac{1}{2}}(2J-1)$ and $v_{J-\frac{1}{2}}(2J-1)$ are expressed by these formulae respectively

$$[2J]_q([2J]_q - [2J-1]_q)u_J(2J-2) \otimes u_{\frac{1}{2}}(1) + q^{4J}u_J(2J) \otimes u_{\frac{1}{2}}(-1) - q^{2J}\left(1 - \frac{1}{q}\right)[2J]_q u_J(2J-1) \otimes u_{\frac{1}{2}}(0) \quad \text{and}$$

$$u_J(2J-2) \otimes u_{\frac{1}{2}}(1) - q^{-1}u_J(2J) \otimes u_{\frac{1}{2}}(-1) + q^{-1}u_J(2J-1) \otimes u_{\frac{1}{2}}(0), \tag{A.5}$$

and we make sure that two vectors become linearly dependent, for $q^N = -1, N = 4J+1$ is odd (55):

$$\left\{v_{J+\frac{1}{2}}(2J-1), \ldots, v_{J+\frac{1}{2}}(-2J+1)\right\} = \left\{v_{J-\frac{1}{2}}(2J-1), \ldots, v_{J-\frac{1}{2}}(-2J+1)\right\}. \tag{A.6}$$

One can construct representations v', demanding $e \cdot v'(2J) = v_{j_2}(2J+1)$ for the first case (54) and (A.4) and $e \cdot v'(2J-1) = v_{j_2}(2J)$ for the second case (55) and (A.6). The solutions to these equations are not unique (the solutions to the homogeneous equations, i.e. $v_{J+\frac{1}{2}}(2J)$ and $v_{J+\frac{1}{2}}(2J-1)$, can be added). The remaining vectors of $\mathcal{I}$ is possible to construct by the action of the lowering generator f on the vectors $v'(2J)$ or $v'(2J-1)$. In the case (54) the solution can be taken in the following form:

$$v'(2J) = u_J(2J-1) \otimes u_{\frac{1}{2}}(1), \qquad v'(h') = f^{2J-h}v'(2J), \quad h' = 2J-1, \ldots, -2J. \tag{A.7}$$

The resulting representation $\{v_{J+1/2}(h), v'(h')\}$, with $h = 2J+1, \ldots, -2J-1, h' = 2J, \ldots, -2J$, consists of the indecomposable representation $\mathcal{I}^{(8J+4)}_{\{4J+3, 4j_{\max}+1\}}$. This is in the agreement with formula (64), as $J = (N-1)/2$.

In the second case (55) and (A.6), one can check that the vectors $\{v_{J+1/2}(h), v'(h')\}, h = 2J+1, \ldots, -2J-1, h' = 2J-1, \ldots, -2J+1$, are forming indecomposable representation $\mathcal{I}^{(8J+2)}_{\{4J+3, 4J-1\}}$ (see (65)), with the following vectors $v'(h')$:

$$v'(2J-1) = qu_J(2J-2) \otimes u_{\frac{1}{2}}(1) + u_J(2J) \otimes u_{\frac{1}{2}}(-1), \qquad v'(h') = f^{2J-h}v'(2J-1). \tag{A.8}$$

Here $h' = 2J-2, \ldots, -2J+1$. By using the algebra relations, it is easy to check that representations $\mathcal{I}^{(8J+4)}_{\{4J+3, 4J+1\}}$ and $\mathcal{I}^{(8J+2)}_{\{4J+3, 4J-1\}}$ are satisfying (56), and to obtain the coefficients $\beta, \gamma, \bar{\beta}, \bar{\gamma}, \tilde{\beta}, \tilde{\gamma}$.

The case of (67). The representations $\mathcal{I}^{(p\mathcal{R})}, p > 1$, appear in the tensor product decompositions $\otimes^n V_3$ quite similarly as $\mathcal{I}^{(\mathcal{R})}$, but now due to the deformation of a sum $V_{4j_1+1} \oplus V_{4j_2+1}$ in the decomposition, with $j_2 > j_1 > J$ (as $4j_2 + 4j_1 + 2 = p\mathcal{R}$, see (61)). It can be checked analogously to the previous case, using Clebsh–Gordan coefficients, taking now tensor product $\bar{V}_{4j+1} \otimes V_3$ $(j > J)$. The difference here is that there are another highest and lowest weights also in the representation $\bar{V}_{4j_2+1}$ besides the weights $\pm j_{1(2)}$, as $\Delta(f^{\mathcal{N}})$ (10) (as well as the operator $\Delta(e^{\mathcal{N}})$) vanishes when $(-q)^{\mathcal{N}} = 1$, and now $\mathcal{N} \leqslant 4j_1$. In the paragraph after (66) we have denoted them as $j_2^i, i = 1, \ldots, p = [(4j_2+1)/\mathcal{N}]$.

In the same way, as above, at the values of q defined by (58) a linear dependence is established between the vectors $f^n v_{j_2}(2j_2^i), 0 \leqslant n < \mathbf{r_p}, i = 1, \ldots, p$ (note that $j_2^p = j_1$), of $\bar{V}_{4j_2+1}$ and the corresponding vectors of $\bar{V}_{4j_1+1}$ with the same h. Solving the equations $ev'(2j_2^i) = v_{j_2}(2j_2^i + 1)$, one constructs all v'-vectors, $v'(2j_2^i - n) = f^n v'(2j_2^i)$, $0 \leqslant n < \mathbf{r_p}$.

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